

ON THE PAIR OF GENERALIZED RULED SURFACES UNDER THE HOMOTHETIC MOTIONS OF E^n

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The purpose of this paper, after presenting a summary of known results of the homothetic motions and the generalized ruled surfaces of n -dimensional Euclidean space E^n , is to define the pair of the fixed and moving generalized ruled surfaces under the homothetic motion and to give some results about the parameter of distributions, apex segments, and apex angles of the pair.

I. Homothetic Motions in E^n

The homothetic motion of a body in n -dimensional Euclidean space is generated by the transformation

$$(1) \quad \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} S & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}, \quad S = hB$$

where B is a proper orthogonal $n \times n$ matrix such that if the transpose of B is denoted by B^T and the unit matrix I_n then $B^T B = I_n$, h is real scalar matrix. The homothetic scale h and the elements of B and c are continuously differentiable functions of a real parameter $t \in I$; x and \bar{x} correspond to the position vectors of the same point with respect to the rectangular coordinate frames of the moving space E and the fixed space \bar{E} , respectively. To avoid the case of affine transformation we assume that $h = h(t) \neq \text{constant}$.

The equation (1) by differentiation with respect to t yields

$$(2) \quad \dot{x} = \dot{S}\bar{x} + \dot{c} + S\dot{\bar{x}}, \quad \bar{x} \in \bar{E}$$

where \dot{x} is the absolute velocity (absolutgeschwindigkeit), $S\bar{x} + \dot{c}$ is the sliding velocity (fuhrungsgeschwindigkeit), and $\dot{S}\bar{x}$ is the relative velocity (relativgeschwindigkeit). For the common fixed points $x \in E$ and $\bar{x} \in \bar{E}$, from the equation (2) we have

$$(3) \quad \dot{S}\bar{x} + \dot{c} = 0.$$

In order to find the position at the initial time of these points we must solve the equation (3). H.Hacısalihoğlu proved in [5] that

$$(4) \quad |S(t)| \neq 0, \forall t \text{ and } \forall n.$$

Therefore we get a differentiable curve α of poles in the fixed space E , called the fixed pole curve. By (1) is uniquely determined the moving pole curve $\bar{\alpha}$ from the fixed pole curve point to point on $\alpha(t) = S\bar{\alpha}(t) + c$. That is, the equations of α and $\bar{\alpha}$ are

$$(5) \quad \alpha = S\bar{\alpha} + c,$$

$$(6) \quad \bar{\alpha} = -\dot{S}^{-1} \dot{c}.$$

II. Generalized Ruled Surfaces

In any k -dimensional generator $E_k(t) = \text{Sp} \{e_1(t), \dots, e_k(t)\}$ of $(k+1)$ -dimensional generalized ruled surface $\varphi \subset E^n$ there exists a maximal linear subspace $K_{k-m}(t) \subset E_k(t)$ with the property in every point of $K_{k-m}(t)$ no tangent space of φ is determined ($K_{k-m}(t)$ contains all singularities of φ in $E_k(t)$) or there exists a maximal linear subspace $Z_{k-m}(t)$ with the property that in every point of $Z_{k-m}(t)$ the tangent space of φ is orthogonal to the asymptotic bundle $A(t) = \{e_1(t), \dots, e_k(t), \dot{e}_1(t), \dots, \dot{e}_k(t)\}$ of the tangent spaces in the points of infinity of $E_k(t)$ (all points of $Z_{k-m}(t)$ have the same tangent space of φ) [2]. We call $K_{k-m}(t)$ the edge space and $Z_{k-m}(t)$ the central space in $E_k(t) \subset \varphi$ [3]. A point of $Z_{k-m}(t)$ is called a central point. If φ possesses generators all of the same type the edge spaces resp. the central spaces generate a generalized ruled surface contained in φ which we call the edge ruled resp. central ruled surface [3]. For $m = k$ the edge ruled surface degenerates in the edge of φ , the central ruled surface in the line of striction. So the ruled surfaces with edge ruled surface generalize the tangent surfaces of E^3 , the ruled surfaces with central ruled surface the ruled surfaces with line of striction of E^3 .

φ has following parameter representation

$$(7) \quad \varphi(t, u_1, \dots, u_k) = \alpha(t) + \sum_{i=1}^k u_i e_i(t), \quad u_i \in \mathbb{R}, \quad t \in I.$$

It is shown in [3] that there exists a distinguished moving orthonormal frame (ONF) of φ $\{e_1, \dots, e_k\}$ with the properties:

i) $\{e_1, \dots, e_k\}$ is an ONF of the $E_k(t) \subset \varphi$,

ii) $\{e_{m+1}, \dots, e_k\}$ is an ONF of the $K_{k-m}(t)$ resp. $Z_{k-m}(t) \subset E_k(t)$,

$$(8) \quad \text{iii) } \dot{e}_i = \sum_{j=1}^k \beta_{ij} e_j + K_i a_{k+i}, \quad 1 \leq i \leq m, \quad \beta_{ij} = -\beta_{ji}, \quad K_i > 0,$$

$$(9) \quad \dot{e}_{m+r} = \sum_{j=1}^m \beta_{(m+r)j} e_j, \quad 1 \leq r \leq k-m,$$

iv) $\{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}\}$ is an ONF.

A moving ONF of φ with the properties i. - iv. is called a principal frame of φ [2].

A leading curve $\alpha \subset E^n$ of a generalized ruled surface φ a leading curve of the edge resp. central ruled surface $\Omega \subset \varphi$ too iff its tangent vector has the form

$$(10) \quad \dot{\alpha} = \sum_{i=1}^k \xi_i e_i + \eta_{m+1} a_{k+m+1}$$

where $\eta_{m+1} \neq 0$, a_{k+m+1} is a unit vector well defined up to the sign with the property that $\{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}, a_{k+m+1}\}$ is an ONF of the tangent bundle $T(t)$ of φ . One shows: $\eta_{m+1} = 0$ in $t \in I$ iff the generator $E_k(t) \subset \varphi$ contains the edge space $K_{k-m}(t)$.

If $\eta_{m+1} \neq 0$ we call m magnitudes and

$$(11) \quad p_i = \eta_{m+1} / K_i, \quad 1 \leq i \leq m$$

the principal parameters of distribution [2]. The parameters are direct generalizations of the parameter of distribution of the ruled surfaces in E^3 . A generalized ruled surface with central ruled surface and no principal parameter of distribution ($m = 0$) is a $(k + 1)$ -dimensional cylinder. We generalize the definition in [1] as

$$(12) \quad p = \sqrt[m]{|p_1 \cdots p_m|}$$

which is called the parameter of distribution of a generalized ruled surface [2].

We call m magnitudes and

$$(13) \quad \varphi_i(t, u) = \alpha(t) + u e_i(t), \quad (t, u) \in I \times \mathbb{R}, \quad 1 \leq i \leq m$$

2-dimensional principal ruled surfaces of the generalized ruled surface φ [4]. For the parametrization (7) and positive integer p if we have

$$(14) \quad \varphi(t + p, u_1, \dots, u_k) = \varphi(t, u_1, \dots, u_k)$$

then we call φ closed, where p is the period [4].

In the case $m=k$, we call k magnitudes and

$$(15) \quad L_i = - \int_0^p \zeta_i(t) dt, \quad 1 \leq i \leq k$$

the apex segments of φ [4]. Suppose $\dim T(t) = k + m + 1 = n$. In this case we define m -apex angles of φ . We call m magnitudes and

$$(16) \quad \lambda_i = \int_0^p w_i(t) dt, \quad 1 \leq i \leq m$$

the apex angles of φ [4], where $w_i, 1 \leq i \leq m$, there exist [3]:

$$(17) \quad \left\{ \begin{array}{l} \dot{a}_{k+i} = -K_i e_i + \sum_{j=1}^m \tau_{ij} a_{k+j} + w_i a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{i\lambda} a_{k+m+\lambda}, \\ \dot{a}_{k+m+1} = - \sum_{j=1}^m w_j a_{k+j} - \sum_{\lambda=2}^{n-k-m} \beta_{\lambda} a_{k+m+\lambda}, \\ \dot{a}_{k+m+s} = \sum_{j=1}^m w_{sj} a_{k+j} + \beta_s a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \beta_{s\lambda} a_{k+m+\lambda}, \\ \text{where } 2 \leq s \leq n-k-m. \end{array} \right. \quad 1 \leq i \leq m,$$

In the case $n=k+m+1$, from (17) we have

$$(18) \quad w_i = - \langle \dot{a}_{k+m+1}, a_{k+i} \rangle, \quad 1 \leq i \leq m,$$

$$(19) \quad \dot{a}_{k+m+1} = - \sum_{i=1}^m w_i a_{k+i}.$$

III. The Pairs of Generalized Ruled Surfaces Under The Homothetic Motions

Let $\bar{\alpha} \subset \bar{E}$ and $\alpha \in E$ be moving and fixed pole curves, respectively. Suppose that $\{\bar{e}_1(t), \dots, \bar{e}_k(t)\}$ is an orthonormal vector field system at $\bar{\alpha}(t)$ and $\bar{E}_k(t) = \text{Sp} \{\bar{e}_1(t), \dots, \bar{e}_k(t)\}$. Then $\bar{E}_k(t)$ generates a $(k+1)$ -dimensional ruled surface with the leading curve $\bar{\alpha}$ in the moving space \bar{E} which is called The Moving Ruled Surface $(\bar{\varphi})$. $\bar{\varphi}$ has following parameter representation

$$(20) \quad \varphi(t, \bar{u}_1 \dots \bar{u}_k) = \bar{\alpha}(t) + \sum_{i=1}^k \bar{u}_i \bar{e}_i(t), \quad \bar{u}_i \in \mathbb{R}, \quad t \in I.$$

Let $\{\varepsilon_1(t), \dots, \varepsilon_k(t)\}$ be an orthogonal vector field system satisfying the following equation at the point $\alpha(t)$ in the fixed space E :

$$(21) \quad S(\bar{e}_i) = \varepsilon_i, \quad 1 \leq i \leq k.$$

If we set

$$(22) \quad \varepsilon_i = h e_i, \quad 1 \leq i \leq k,$$

then we get orthonormal vector field system $\{e_1, \dots, e_k\}$ and therefore $E_k(t) = \text{Sp} \{e_1, \dots, e_k\}$ generates a $(k+1)$ -dimensional ruled surface with leading curve is given by (1) in the fixed space E which is called The Fixed Ruled Surface (φ) .

From the equations (21), (22), and $S = hB$ we get

$$(23) \quad B\bar{e}_i = e_i, \quad 1 \leq i \leq k.$$

On the other hand, for ONF $\{\bar{e}_1, \dots, \bar{e}_k, \bar{a}_{k+1}, \dots, \bar{a}_{k+m}\}$ of asymptotic bundle $\bar{A}(t)$ of $\bar{\varphi}$, and $S\bar{a}_{k+i} = A_{k+i}, 1 \leq i \leq m, \{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}\}$ is an ONF of the asymptotic bundle $A(t)$ of φ , where

$$(24) \quad A_{k+i} = h a_{k+i}, \quad 1 \leq i \leq m.$$

And also, in the case of $\dim \bar{T}(t) = k+m+1$, we have the results:

$$(25) \quad A_{k+m+1} = h a_{k+m+1} = S\bar{a}_{k+m+1},$$

$$(26) \quad \bar{a}_{k+m+1} = - \sum_{i=1}^m w_i a_{k+i}.$$

Theorem III.1.. Let φ and $\bar{\varphi}$ be the $(k+1)$ -dimensional fixed and moving generalized ruled surfaces with the leading curves α and $\bar{\alpha}$,

respectively. If $\{e_1, \dots, e_k\}$ and $\{\bar{e}_1, \dots, \bar{e}_k\}$ are the ONFs of φ and $\bar{\varphi}$, then we have the following results:

$$(27) \quad i) \beta_{ij} = \bar{\beta}_{ij}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq k \quad (i \neq j),$$

$$(28) \quad \text{if } i=j, \text{ then } (\dot{h}/h) + \beta_{ii} = \dot{S}S^{-1} + \bar{\beta}_{ii}, \quad 1 \leq i \leq m,$$

$$(29) \quad \text{ii) } K_i = \bar{K}_i, \quad 1 \leq i \leq m.$$

Proof. From (22) by differentiation we observe that

$$\dot{\varepsilon}_i = \dot{h}\varepsilon_i + h\dot{\varepsilon}_i,$$

$$\dot{\varepsilon}_i = (\dot{h}h^{-1}) \varepsilon_i + h \left(\sum_{j=1}^k \beta_{ij} e_j + K_i a_{k+i} \right),$$

$$\dot{\varepsilon}_i = (\dot{h}h^{-1}) \varepsilon_i + \sum_{j=1}^k B_{ij} e_j + K_i A_{k+i}, \quad 1 \leq i \leq m,$$

$$(30) \quad \dot{\varepsilon}_i = (\dot{h}h^{-1} + \beta_{ii}) \varepsilon_i + \sum_{j=1, j \neq i}^k \beta_{ij} e_j + K_i A_{k+i}.$$

On the other hand, from (21) by differentiation we observe that

$$\dot{\bar{\varepsilon}}_i = \dot{S}\bar{\varepsilon}_i + S\dot{\bar{\varepsilon}}_i,$$

$$\dot{\bar{\varepsilon}}_i = \dot{S}S^{-1} \bar{\varepsilon}_i + S \left(\sum_{j=1}^k \bar{\beta}_{ij} \bar{e}_j + \bar{K}_i \bar{a}_{k+i} \right),$$

$$\dot{\bar{\varepsilon}}_i = \dot{S}S^{-1} \bar{\varepsilon}_i + \sum_{j=1}^k \bar{\beta}_{ij} \bar{e}_j + \bar{K}_i A_{k+i}, \quad i \leq i \leq m,$$

$$(31) \quad \dot{\bar{\varepsilon}}_i = (\dot{S}S^{-1} + \bar{\beta}_{ii}) \bar{\varepsilon}_i + \sum_{j=1, j \neq i}^k \bar{\beta}_{ij} \bar{e}_j + \bar{K}_i A_{k+i}.$$

If we consider the equation (30) together with the equation (31), the theorem is proved.

Theorem III.2. We have the following results:

$$(32) \quad h \bar{\xi}_i = \xi_i, \quad 1 \leq i \leq k,$$

$$(33) \quad |h \bar{\eta}_{m+1}| = |\eta_{m+1}|,$$

where,

$$(34) \quad \dot{\bar{\alpha}} = \sum_{i=1}^k \bar{\xi}_i \dot{\bar{e}}_i + \bar{\eta}_{m+1} \dot{\bar{a}}_{k+m+1},$$

$$(35) \quad \dot{\alpha} = \sum_{i=1}^k \xi_i e_i + \eta_{m+1} a_{k+m+1} .$$

Proof. From (34) we get

$$S\dot{\bar{\alpha}} = S \left(\sum_{i=1}^k \xi_i \bar{e}_i + \bar{\eta}_{m+1} \bar{a}_{k+m+1} \right),$$

$$\dot{\alpha} = \sum_{i=1}^k \bar{\xi}_i S\bar{e}_i + \bar{\eta}_{m+1} S\bar{a}_{k+m+1} \quad (S\bar{\alpha} = \dot{\alpha}),$$

and using equations (21), (25), and (22) we observe that

$$(36) \quad \dot{\alpha} = \sum_{i=1}^k h \bar{\xi}_i e_i + h \bar{\eta}_{m+1} a_{k+m+1} .$$

Thus, if we consider the equation (35) together with the equation (36), the theorem is proved.

From these theorems we get the following corollaries.

Corollary 1.

$$(37) \quad |p_i| = |h| |\bar{p}_i|, \quad 1 \leq i \leq m.$$

Corollary 2.

$$(38) \quad p = |h| \bar{p} .$$

Corollary 3. For the apex segments L_i and \bar{L}_i of the fixed and moving generalized ruled surfaces φ and $\bar{\varphi}$ under the closed homothetic motion $x=S\bar{x}+c$, respectively, we have

$$(39) \quad dL_i = h d\bar{L}_i, \quad 1 \leq i \leq m=k.$$

Proof.

$$L_i = - \int_0^p \xi_i(t) dt, \quad 1 \leq i \leq m=k,$$

$$dL_i = - \xi_i(t) dt.$$

$$\bar{L}_i = - \int_0^p \bar{\xi}_i(t) dt, \quad 1 \leq i \leq m=k,$$

$$d\bar{L}_i = - \bar{\xi}_i(t) dt,$$

and using (32), we get (39).

Theorem III.3. For the apex angles λ_i and $\bar{\lambda}_j$ of the fixed and moving generalized ruled surfaces φ and $\bar{\varphi}$ under the closed homothetic motion

$$x = S\bar{x} + c,$$

respectively, we have

$$(40) \quad \lambda_i = \bar{\lambda}_i + \int_0^p \langle B\bar{a}_{k+i}, \dot{B}\bar{a}_{k+m+1} \rangle, \quad 1 \leq i \leq m.$$

Proof. From (35) we get

$$a_{k+m+1} = B\bar{a}_{k+m+1},$$

$$\dot{a}_{k+m+1} = \dot{B}\bar{a}_{k+m+1} + B\dot{\bar{a}}_{k+m+1},$$

and using (19) and (26) we have

$$-w_i = -\bar{w}_i + \langle B\bar{a}_{k+i}, \dot{B}\bar{a}_{k+m+1} \rangle, \quad 1 \leq i \leq m.$$

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