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ON THE GENERALIZED DARBOUX CURVES

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ABSTRACT

G. Saban studied on Darboux curves and obtained some results on a surface in E^3 [1]. E. Özdamar and H. Hacısalihoğlu evaluated the center of (n-1)-osculating sphere at α (t) of a curve a in E^n [2].

In this paper we defined general Darboux curve on a hypersufrace M in E^n. Then we generalized some results of G. Saban by using [2].

I. INTRODUCTION

This section includes some basic concepts and definitions about curves. Also, the theorem of Özdamar and Hacısalihoğlu will be given here.

Let α be a curve in E^n and $\{V_1, V_2, \ldots, V_n\}$ be the system of Frenet vector fields of α. Then i-th curvature of α is $k_i(s)$ and

$$k_i(s) = \langle V_i'(s), V_{i+1}(s) \rangle$$

where $1 \leq i < n$ and "′" denotes d / ds [3], s denotes the arc-length of α.

Frenet formulas is known [3] as

$$V'_1 = D_{V_1} V_1 = k_1 V_2,$$

$$V'_i = D_{V_1} V_i = -k_{i-1} V_{i-1} + k_i V_{i+1},$$

$$V'_n = D_{V_1} V_n = -k_{n-1} V_{n-1}.$$

Now let α be a curve on a hypersurface M in E^n. If X_1 is the unit tangent vector field of α and N is the unit normal vector field of M, then we can find the system of vector fields $\{X_1, X_2, \ldots, X_{n-1}, N\}$ which is called natural frame field system of the pair of (α, M) [4]. And we know that the i-th geodesic curvature function of α is

$$k_{ig}(s) = < X'_i(s), X_{i+1}(s) > , 1 \leq i \leq n-1$$

If we use $X_n = N$, then we have the following derivative formulas

$$D_{X_1}X_i = X'_i = -k_{(i-1)g} X_{i-1} + k_{ig}X_{i+1} + II(X_i, X_i) N$$

$$D_{X_1}N = N' = -II(X_1, X_1)X_1 - II(X_1, X_2)X_2 - \dots - II(X_1, X_{n-1})X_{n-1}$$

where II is the second fundamental form of M and $k_{0g} = k_{(n-1)g} = 0$.

I.1. Definition: Let α be a curve on a hypersurface M in E^n. If the tangent space of (n-1)-osculating sphere coincides with tangent space of M at every point α (s), then α is called a generalized Darboux curve on M.

By the definition it is clear that

$$\alpha(S) - a = \lambda N$$

where "a" is the center of (n-1)-osculating sphere and N is the unit normal of M.

In [2] Özdamar and Hacısalihoğlu proved the following theorem about the center of (n-1)-osculating sphere.

I.1. Theorem: Let α be a curve in E^n, k_i be i-th curvature function of α and $k_{n-1} \neq 0$ at α (s) for every s. The center "a" of (n-1)-osculating sphere is

$$\alpha(s) - \sum_{i=2}^{n-1} m_i V_i + \lambda V_n, \; 2 < i < n$$

where $\{V_1, V_2, \dots, V_n\}$ is the system of Frenet n-frame, $m_1 = 0$, $m_2 = -1 / k_1$, $\lambda \varepsilon$ IR and $m_i = \{m'_{i-1} + m_{i-2} k_{i-2}\} 1 / k_{i-1}$ [2].

II. GENERALIZED RESULTS

In this section we will give generalized results, about Darboux curves on a hypersurface M in E^n, resembling Saban's in E^3.

II.1. Theorem: Let a be the center of the (n-1) -osculating sphere at the point α (s) of a curve α in E^n, then α (s)-a and $\alpha'''(s)$ is perpendicular to each other.

Proof: By using Frenet formulas we can show that

$$\alpha'(s) = V_1$$
$$\alpha''(s) = k_1 V_2$$
$$\alpha'''(s) = k'_1 V_2 + k_1 V'_2$$

and thus

$$\alpha'''(s) = -k^2_1 V_1 + k'_1 V_2 + k_1 k_2 V_2.$$

So

$$< \alpha(s)-a, \alpha'''(s) > = < \sum_{i=2}^{n-1} m_i(s) V_i(s) + \lambda V_n(s), -k^2_1(s) V_1(s)$$
$$+ k'_1(s) V_2(s) + k_1(s) k_2(s) V_3(s) >$$
$$= -m_1(s) k^2_1(s) + m_2(s) k'_1(s) + m_3(s) k_1(s) k_2(s)$$
$$= (-1/k_1(s))\, k'_1(s) + [(-1/k_1(s))'1/k_2(s)]\; k_1(s) k_2(s)$$
$$= -\frac{k'_1(s)}{k_1(s)} + \frac{k'_1(s)}{k^2_1(s)} k_1(s)$$
$$= 0$$

which completes the proof of the theorem.

We can evaluate the vector α''' in terms of higher order geodesic curvatures, so we obtain

$$\alpha''' = -k^2_{1g} X_1 + k'_{1g} X_2 + k_{1g} k_{2g} X_3 + [k_{1g} II(X_1, X_2) +$$
$$(II(X_1, X_2))'] N + II(X_1, X_1) N'.$$

Therefore we find

$$< \alpha''', N > = k_{1g} II(X_1, X_2) + (II(X_1, X_1))'.$$

II.1. Definition: Let α be a curve on hypersurface M in E^n, then we call that

$$k_{1g} II(X_1, X_2) + (II(X_1, X_1))'$$

is generalized Darboux function through α and it is denoted by D.

II.2. Theorem: Let M be a hypersurface in E^n and α be a curve in M. If α is a Darboux curve on M then we see

$$D = 0$$

through the α.

Proof: Since α is a Darboux curve on M, by I.1. Definition, we have

$$\alpha(s) - a = \lambda N.$$

Also we defined that

$$D = < \alpha'''(s), N > .$$

And so

$$\begin{aligned} D &= < \alpha'''(s), \lambda_1 (\alpha(s) - a) > \\ &= \lambda_1 < \alpha'''(s), \alpha(s) - a > \\ &= 0. \end{aligned}$$

II.3. Theorem: For all curves on a hyperplane and a hypershere $D = 0$.

Proof: It is clear for hyperplane because of the shape operator $S = 0$.

We know that shape operator is $S = 1/r\, I_{n-1}$ for the hypersphere with radious r. It follows that

$$\begin{aligned} II(X_1, X_1) &= < S(X_1), X_1 > \\ &= 1/r. \end{aligned}$$

Since $1/r$ is a constant we obtain

$$(II(X_1, X_1))' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

And we find that

$$\begin{aligned} II(X_1, X_2) &= < S(X_1), X_2 > \\ &= 1/r < X_1, X_2 > \\ &= 0 \end{aligned}$$

So,

$$II(X_1, X_2) = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

From (1) and (2), we have that

$$D = 0.$$

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