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THE DRALL AND THE SCALAR NORMAL CURVATURE OF (r+1)-DIMENSIONAL GENERALISED RULED SURFACES

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SUMMARY

In this paper, we obtain some relationships between curvatures of (r+1)-dimensional generalised ruled surfaces. We also calculate the drall of a generalized ruled surface when the base curve is taken as an orthogonal trajectory of the generated spaces.

INTRODUCTION

All manifolds, maps, vector fields etc. will be assumed smooth. Let E^n be n-dimensional Euclidean space and M a submanifold of E^n. Let $\bar{D}$ denote the standard Riemannian connection of E^n and let D denote the Riemannian connection of M. For any vector fields X,Y on M we have the Gauss equation.

$$\bar{D}_X Y = D_X Y + V(X,Y) \tag{1.1}$$

where $D_X Y$, V(X,Y) are respectively the tangential, normal components of $\bar{D}_X Y$.V is called the second fundamental form of M. We also have the Weingarten equation giving the tangential and normal components of $\bar{D}_X \xi$, where ξ is a normal vector field on M,

$$\bar{D}_X \xi = - A_\xi(X) + D_X{}^\perp \xi \tag{1.2}$$

Let X,Y be vector field on M, ξ a normal vector field and $<,>$ the standard metric on E^n. From (1.1) we have

$$<\bar{D}_X Y, \xi> = <V(X,Y), \xi> \tag{1.3}$$

and then (1.2) implies

$$<V(X,Y),\xi> = <A_\xi(X),Y> \tag{1.4}$$

Let $\{\xi_1,\ldots,\xi_{n-m}\}$ be an orthonormal basis of $\chi^\perp(M)$, the space of normal vector fields on M. Then there exist smooth functions $V^j(X,Y)$ (j=1,...,n—m) from M into R such that

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$$V(X,Y) = \sum_{j=1}^{n-m} V^j(X,Y)\xi_j \tag{1.5}$$

and furthermore we may define the mean curvature vector field H by

$$H = \sum_{j=1}^{n-m} (\text{trace } A_{\xi_j}/m)\xi_j \tag{1.6}$$

and the mean curvature function as $\| H \|$. At a point $p \in M$, $H(p)$ is called the mean curvature vector and $\| H(p) \|$ the mean curvature at p [1].

If, for each $p \in M$, $H(p) = 0$, then M is said to be minimal [1].

Let ξ be a unit normal vector, then the Lipschitz-Killing curvature in the direction ξ at the point $p \in M$ is defined by [2]:

$$G(p,\xi) = \det A_\xi(p)\,. \tag{1.7}$$

The Gauss curvature is defined by

$$G(p) = \sum_{j=1}^{n-m} G(p,\xi_j) \tag{1.8}$$

and if $G(p) = 0$ for all $p \in M$, we say M is developable. In particular, if the Lipschitz-Killing curvature is zero for each point and each normal direction, then M is developable.

Following [3], we define M(A) for any symmetric matrix $A = [a_{ij}]$ by

$$M(A) = \sum_{i,j} (a_{ij})^2\,. \tag{1.9}$$

Let I be an open interval and $\alpha: I \to E^n$ a curve in Euclidean space. For each $t \in I$, let $\{e_1(t), \ldots, e_r(t)\}$ $(1 \leq r \leq n-2)$ be an orthonormal set of vectors spanning the r-dimensional subspace $W_r(t)$ of $T_{\alpha(t)}E^n$. We have

$$\langle e_i, e_j \rangle = \delta_{ij} \quad (i,j = 1, \ldots r) \tag{1.10}$$

and denoting by $\dot{e}_i$ the derivative of the vector field e_i along the curve α;

$$\langle \dot{e}_i, e_j \rangle + \langle e_i, \dot{e}_j \rangle = 0 \quad (i,j = 1, \ldots r) \tag{1.11}$$

We may define an $(r+1)$–dimensional submanifold M of E^n as follows.

Definition 1.1.

Let α, $\{e_i\}$ be as above and define φ: $IxE^r \to E^n$ by

$$\varphi(t,u_1,\ldots,u_r) = \alpha(t) + \sum_{i=1}^{r} u_i e_i(t) \tag{1.12}$$

for all $(t,u_1,\ldots,u_r) \in IxE^r$. Let $M=\varphi(G)$ where $G=IxE^r \subseteq E^{r+1}$. Note that

$$\mathrm{rank}(\varphi_t,\varphi_{u_1},\ldots,\varphi_{u_r}) = \mathrm{rank}(\alpha(t) + \sum_{i=1}^{r} u_i e_i(t), e_1(t),\ldots,e_r(t)) = r+1$$

so M is an (r+1)–dimensional submanifold of E^n. We call M an (r+1)–dimensional generalised ruled surface. The curve α is called the base curve of the generalised ruled surface and the subspace $W_r(t)$ is called the generating space (or briefly, the generation) at the point $\alpha(t)$ [4].

Definition 1.2.

The subspace A(t) given by

$$A(t) = \mathrm{Sp}\{e_1(t),\ldots,e_r(t), \dot{e}_1(t),\ldots,\dot{e}_r(t)\} \tag{1.13}$$

with dimension $\dim A(t) = r + m$, $0 \leq m \leq r$, is said to be the asymptotic bundle of the generalised ruled surface.

$W_r(t)$ is a subspace of A(t) and, using the Gram-schmidt orthogonalisation process, basis of the form:

$$\{e_1(t),\ldots e_r(t), a_{r+1},\ldots,a_{r+m}\} \tag{1.14}$$

may be found. Then there exist b_{ij}, c_{ik} such that

$$e_i = \sum_{j=1}^{r} b_{ij}e_j + \sum_{k=1}^{m} c_{ik}a_{r+k}, \; (i = 1 \ldots r), \tag{1.15}$$

with $b_{ij} = - b_{ji}$ by (1.11). The basis $\{e_1(t),\ldots,e_r(t)\}$ of $W_r(t)$ uniquely determines the basis of the asymptotic bundle of a generalised ruled surface and $\{e_1(r),\ldots,e_r(t)\}$ is called the natural carrier basis of $W_r(t)$ [4].

Now let $\eta_{m+1} = \langle \alpha_r(t), a_{r+m+1} \rangle$, $K_k = \langle \dot{e}_k(t), a_{r+k} \rangle$ for $k=1,\ldots,m$,

so that $\dot{e}_i = \sum_{j=1}^{r} b_{ij}e_j + K_i a_{r+i}, (1 \leq i \leq m, K_i > 0)$, $\dot{e}_i = \sum_{j=1}^{r} b_{ij}e_j$

$(m<i\leq r)$. We now define the following:

$$\delta_k = \eta_{m_1}/K_k \quad (k=1,\dots,m) \tag{1.16}$$

and note that each δ_k is invariant under a reparameterisation $t \to t^*$ with $dt/dt^* > 0$. δ_k is called the k^{th} principle drall (principal distribution parameter) of M lying in $W_r(t)$ [4]. The drall (distribution parameter) of M is defined by

$$\delta = |\delta_1 \dots \delta_m|^{1/m} \tag{1.17}$$

We remark that the k^{th} principle drall and the drall are equal for a ruled surface with $m = 1$ in E^3.

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Let M be an $(r+1)$-dimensional generalised ruled surface and s the arc length parameter of the curve α. Let $\{e_1(s),\dots,e_r(s)\}$ be an orthonormal basis of the generating space $W_r(s)$. Let us choose the base curve α to be an orthogonal trajectory of the generating spaces $W_r(s)$. M is given by

$$\varphi(s,u_1,\dots,u_r) = \alpha(s) + \sum_{i=1}^{r} u_i e_i(s), \; u_i \in R \tag{2.1}$$

Let $\{e_0, e_1,\dots,e_r\}$ be a (local) orthonormal basis of the space of vector fields $\chi(M)$ and let us choose $e_0 = \varphi^* (\partial/\partial s)$. By (2.1),

$$\varphi_s = \dot{\alpha}(s) + \sum_{i=1}^{r} u_i \dot{e}_i(s), \; \varphi_{u_i} = e_i(s) \tag{2.2}$$

then

$$\bar{D}_{e_i} e_j = 0 \quad (i,j = 1,\dots,r) \tag{2.3}$$

and using (1.1),

$$V(e_i,e_j) = 0 \quad (i,j = 1,\dots,r) \tag{2.4}$$

and since $\bar{D}_{e_i}e_0 \perp e_j$ and $\bar{D}_{e_i}e_0 \perp e_0$ (for each i,j), then

$$\bar{D}_{e_i}e_0 = V(e_i,e_0) \quad (i=1,\dots,r) \tag{2.5}$$

Let $\{\xi_1,\dots,\xi_{n-r-1}\}$ be an orthonormal basis of normal vector fields. Then $\{e_0,e_1,\dots,e_r,\xi_1,\dots,\xi_{n-r-1}\}$ gives a basis of $T_\varphi E^n$ for each point $p \in M$. Let us write

$$\bar{D}e_o\xi_j = a^j_{oo}e_o + \sum_{t=1}^{r} a^j_{ot}e_t + \sum_{q=1}^{n-r-1} b^j_{oq}\xi_q \quad (j=1,\ldots,n-r-1)$$

$$\bar{D}e_i\xi_j = a^j_{oi}e_o + \sum_{r=1}^{r} a^j_{it}e_t + \sum_{q=1}^{n-r-1} b^j_{iq}\xi_q \quad (i=1,\ldots,r) \tag{2.6}$$

Where the a^j_{it} are coefficients of the matrix of $A_{\xi j}$:

$$A_{\xi j} = - \begin{vmatrix} a^j_{oo} & a^j_{o1} & \ldots & a^j_{or} \\ a^j_{o1} & a^j_{11} & \ldots & a^j_{1r} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & & \ldots & \cdot \\ a^j_{or} & a^j_{r1} & \ldots & a^j_{rr} \end{vmatrix} \quad (j=1,\ldots,n\text{-}r\text{-}1) \tag{2.7}$$

This matrix simplifies since, using (2.6), $<\bar{D}_{e_i}e_t, \xi_j> = -a^j_{it}$ $(i,t=1,\ldots,r;j=1,\ldots,n-r-1)$, and then by (2.3), $a^j_{it}=0$, and now we may write (2.7) as

$$A_{\xi j} = - \begin{vmatrix} a^j_{oo} & a^j_{o1} & \ldots & a^j_{or} \\ a^j_{o1} & 0 & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ a^j_{or} & 0 & \ldots & 0 \end{vmatrix} \tag{2.8}$$

furthermor

furthermore, (2.6) and (1.4) lead respectively to the relations:

$$<\bar{D}_{e_i}e_o,\xi_j> = -a^j_{oi}, \quad (i=1,\ldots,r;j=1,\ldots,n-r-1)$$

and

$$<V(e_i,e_o),\xi_j> = <A_{\xi j}(e_i),e_o)> = -a^j_{oi}, \quad (1\leq i\leq r;\ 1\leq j\leq n-r-1),$$

and therefore, by (1.5) and (2.5);

$$V(e_ie_o) = \bar{D}_{e_i}e_o = -\sum_{j=1}^{n-r-1} a^j_{oi}\xi_j \quad (i=1,\ldots,r) \tag{2.9}$$

Now let X,Y be vector fields on the m-dimensional Riemannian manifold M whose curvature tensor field is R. As in [6] we have

$$<X,R(X,Y)Y> = <V(X,X),V(Y,Y)> - <V(X,Y),V(X,Y)> \tag{2.10}$$

where V is the 2^{nd} fundamental form of M embedded in E^n.

Definition 2.2.

Let M be any m-dimensional Riemannian manifold with curvature tensor R. Let $\{e_1,\ldots,e_m\}$ be an orthonormal basis of T_pM, $p \in M$. Then the Ricci curvature tensor field S is defined by (see [7]):

$$S(p):T_pM\times T_pM\to R;\ (X,Y)\to S(p)(X,Y) = \sum_{i=1}^{m} <R(e_i,X)Y,e_i> \qquad (2.11)$$

The scalar curvature of M is defined by ([7]);

$$r(p) = \sum_{i=1}^{m} S(p)(e_i,e_i) \qquad (2.12)$$

or, by (2.11),

$$r(p) = \sum_{i=1}^{m} \sum_{j=1}^{m} <R(e_j,e_i)e_i,e_j> \qquad (2.13)$$

In order to calculate the Ricci curvature of M in the direction of the vector fields $e_t(t=1,\ldots,r)$, we use (2.4), (2.9), (2.10) and (2.11) to obtain

$$S(e_t,e_t) = \sum_{j=1}^{n-r-1} (a^j{}_{ot}),\ (t=1,\ldots,r) \qquad (2.14)$$

and, for the direction e_o;

$$S(e_o,e_o) = -\sum_{t=1}^{r} \sum_{j=1}^{n-r-1} (a^j{}_{ot})^2 \qquad (2.15)$$

so that, from (2.14) and (2.15),

$$S(e_o,e_o) = \sum_{t=1}^{r} S(e_t,e_t)$$

now we have proved the following:

Theorem 2.1.

Let $\{e_1,\ldots,e_r\}$ be an orthonormal basis of the generating space of the (r+1)-dimensional generalised ruled surface M and $\{e_o,e_1,\ldots,e_r\}$ an orthonormal basis of $\chi(M)$. If the base curve of M is chosen as an orthonormal trajectory of the generating space, then the Ricci curvature in the direction of e_0 is equal to the sum of the Ricci curvatures in the directions of the vector fields forming a basis of the generating space.

By (2.12), the scalar curvature of the (r+1)-dimensional generalised ruled surface M may be expressed as

$$r = S(e_0,e_0) + \sum_{t=1}^{r} S(e_t,e_t) = 2S(e_0,e_0),$$

so we have the following corollary:

Corollary 2.1.

If an (r+1)-dimensional generalised ruled surface M has an orthogonal trajectory of the generating space chosen as base curve, then the scalar curvature of M is equal to twice the Ricci curvature in the direction of the vector field e_0.

(2.15) may now be written in the following way:

$$r = -2 \sum_{j=1}^{n-r-1} \sum_{t=1}^{r} (a^j_{ot})^2 \qquad (2.16)$$

and using (1.3), (1.4), (1.5), (2.6) and (2.8) we have:

$$V(e_0,e_0) = \sum_{j=1}^{n-r-1} (\text{trace } A_{\xi_j})\xi_j \qquad (2.17)$$

and now (1.6) gives;

$$H = (1/r+1)\ V(e_0,e_0).$$

If M is minimal then H is zero and so

$$V(e_0,e_0) = 0. \qquad (2.18)$$

We say that $X_p, Y_p \in T_p M$ are conjugate if $V(X_p, Y_p) = 0$, [5]. We have the following theorem:

Theorem 2.2.

Let $\{e_1,\dots,e_r\}$ be an orthonormal basis for the generating space of an (r+1)-dimensional generalised ruled surface M, and let e_0 be a unit tangent vector field to the base curve, the letter taken to be an orthogonal trajector of the generating space of M. Then the ruled surface M is totally geodesic iff e_0 is conjugate to each vector e_i, $i=1,\dots,r$.

Proof:

$\{e_0,e_1,\ldots,e_r\}$ is an orthonormal basis of $\chi(M)$ and for each X,Y $\in \chi(M)$ we may write

$$X=a_0e_0+\sum_{i=1}^{r} a_ie_i, \quad Y=b_0e_0+\sum_{i=1}^{r} b_ie_i$$

and then

$$V(X,Y)=a_0b_0V(e_0,e_0)+\sum_{i=1}^{r}(a_ib_0+a_0b_i)V(e_i,e_0)+\sum_{i=1}^{r}a_ib_iV(e_i,e_i) \tag{2.19}$$

$:\Rightarrow$

If M is totally geodesic, then V is identically zero ([7]), so e_0 is certinly conjugate to e_i, $i=1,\ldots,r$.

$:\Leftarrow$

If $v(e_0,e_i)=0$ for $i=1,\ldots,r$, then by (2.4) and (2.18), (2.19) reduces to $V(X,Y)=0$, and this completes the proof of the theorem.

If trace $A_{\xi_j}=-a^j_{00}$ from (2.8) is subsituted into (1.6) we obtain

$$(r+1)^2\|H\|^2=\sum_{j=1}^{n-r-1}(a^j_{00})^2 \tag{2.20}$$

Definition 2.3.

Let $\{\xi_1,\ldots,\xi_{n-m}\}$ be an orthonormal basis of $\chi^{\perp}(M)$. Then the scalar normal curvature K_N of M is defined by([3]);

$$K_N=\sum_{i,j=1}^{n-m}M(A_{\xi_i}A_{\xi_j}-A_{\xi_j}A_{\xi_i}) \tag{2.21}$$

Theorem 2.3.

The scalar normal curvature of an (r+1)-dimensional ruled surface M is

$$K_N=2\{[(r+1)^2\|H\|^2-(1/2)r]^2-\sum_{i,j=1}^{n-r-1}\sum_{k=0}^{r}a^i_{0t}a^j_{0k}a^j_{0t}a^i_{0k}\}, \tag{2.22}$$

where $\|H\|$ and r are the mean curvature vector field and scalar curvature of M respectively, and the a^j_{0k}'s are the elements of the matrix A_{ξ_j}.

Proof:

We have (2.8)

$$A_{\xi_i} = - \begin{vmatrix} a^i_{00} & a^i_{01} & \dots & a^i_{0r} \\ a^i_{01} & 0 & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a^i_{0r} & 0 & \dots & 0 \end{vmatrix}$$

and similarly for A_{ξ_j}. We may now compute $A_{\xi_i}A_{\xi_j}-A_{\xi_j}A_{\xi_i}$:

$$A_{\xi_i}A_{\xi_j}-A_{\xi_j}A_{\xi_i}=[b_{tk}]=[a^i_{0t}a^j_{0k}-a^j_{0t}a^i_{0k}] \qquad (2.23)$$

(where $t,k=0,1,\dots,r$; $i,j=1,\dots,n-r-1$). Then by (1.9), we obtain

$$M(A_{\xi_i}A_{\xi_j}-A_{\xi_j}A_{\xi_i}) = \sum_{t,k=0}^{r} (b_{tk})^2 = \sum_{t,k=0}^{r} (a^i_{0t}a^j_{0k}-a^j_{0t}a^i_{0k})^2. \qquad (2.24)$$

($i,j=1,\dots,n-r-1$) and so, by (2.21);

$$K_N = \sum_{i,j=1}^{n-r-1} \sum_{t,k=0}^{r} (a^i_{0t}a^j_{0k}-a^j_{0t}a^i_{0k})^2 \qquad (2.25)$$

and by expanding and using (2.16), (2.20) we complete the proof.

Corollary 2.2.

The scalar normal curvature of a minimal $(r+1)$-dimensional generalised ruled surface is given by:

$$K_N=2((r^2/4) - \sum_{i,j=1}^{n-r-1} \sum_{t,k=1}^{r} a^i_{0t}a^j_{0k}a^j_{0t}a^i_{0k})$$

Proof:

For a minimal surface we have $H=0$ and so the corollary is clear.

Corollary 2.3.

For a totally geodesic $(r+1)$-dimensional generalised ruled surface, the scalar normal curvature is identically zero.

Proof:

M is totally geodesic implies V is identically zero and so A_{ξ_j} is the zero map for each $j=1,\ldots,n-r-1$.

Theorem 2.4.

Let M be an (r+1)-dimensional generalised ruled surface. Let the base curve α be an orthogonal trajectory of the generating space be parameterised by arc length. Then the k^{th} principle distribution parameter is

$$\delta_k=(1-\sum_{t=1}^{r}\eta^2{}_t)^{1/2}/(\|\bar{D}_{e_0}e_k\|^2-\sum_{j=1}^{r}<\bar{D}_{e_0}e_k,e_j>^2)^{1/2}$$

(k=1,...,m), and the distribution parameter (drall) is

$$\delta=(1-\sum_{t=1}^{r}\eta^2{}_t)^{1/2}/\prod_{k=1}^{m}(\|\bar{D}_{e_0}e_k\|^2-\sum_{j=1}^{r}<\bar{D}_{e_0}e_k,e_j>^2)^{1/2m} \quad (k=1,\ldots,m)$$

Proof:

Using (1.16) and (1.17) we obtain

$$\delta=\{\|\dot{\alpha}-\sum_{j=1}^{r}<\dot{\alpha},e_j>e_j-\sum_{t=1}^{r}<\dot{\alpha},a_{r+1}>a_{r+1}\|\}/\{\|\dot{e}_k-\sum_{j=1}^{r}<\dot{e}_k,e_j>e_j\|\}\ (k=1,\ldots,m) \quad (2.26)$$

The base curve α is an orthogonal trajectory so $<\dot{\alpha},e_j>=0$ for $j=1,\ldots,r$. Substituting

$<\dot{\alpha},e_j>=0$ $(j=1,\ldots,r)$ $\dot{e}_k=\bar{D}_{e_0}e_k$ and $<\dot{\alpha},a_{r+1}>=\eta_t$ $(t=1,\ldots,m)$ into (2.26), the desired result is obtained.

REFERENCES

[1] KOBAYASHI, S. and NOMIZU, K., Foundations of differential geomerty, Vols. I, II (1963) Interscience Publishers (John Wiley & sons), New-York.

[2] THAS, C., Properties of ruled surfaces in the Euclidian space E^n, Acedemice Sinica Vol. 6, 1, 133-142.

[3] HOUH, C.S., Surfaces with maximal Lipschitz-Killing curvature in the direction of mean curvature vector. Proc. Amc. Maths. Soc. Vol. 35, 2, 537-542 (1972).

[4] FRANK, H. und GIERING, O., Verallgemeinerte regelflachan. Math. Zelt. 150, 261–271 (1976).

[5] FRANK, H., On kinematics of the n-dimensional Euclidean space. Contributions to geometry. Proceedings of the Geometry Symposium in Siegen. 355–342 (1978).

[6] HACISALİHOĞLU, H.H., Diferensiyel geometri. I.U. Fen-Edebiyat Fakültesi yayınları, Mat.2, Malatya (1983).

[7] CHEN, B.Y., Geometry of manifolds. Marcel Dekkar, New York, (1973).