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ON THE PERIODIC SOLUTIONS OF CERTAIN CLASS OF EIGHTH - ORDER DIFFERENTIAL EQUATIONS

By

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ABSTRACT

The aim of this paper is to investigate sufficient conditions (Theorem 1) for the nonexistence of nontrivial periodic solutions of the autonomous equation (1.1) and (Theorem 2) the existence of periodic solutions of the nonautonomous equation (1.2).

1. INTRODUCTION

In this paper we consider certain eighth order autonomous and nonautonomous differential equations. In [1] and [2] some interesting results are studied for sixth and seventh order differential equations.

We propose to obtain similar results for

$$x^{(8)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + f_5(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}, x^{(6)}, x^{(7)})\dddot{x}$$
$$+ f_6(\dot{x})\ddot{x} + f_7(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}, x^{(6)}, x^{(7)})\dot{x} + f_8(x) = 0 \qquad (1.1)$$

wherein a_2 a_3, a_4 are constants and f_5, f_6, f_7, f_8 are continuous functions depending only on the arguments shown,

and

$$x^{(8)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5\dddot{x} + g_6(\dot{x})\ddot{x} + g_7(\dot{x}, \ddot{x})\dot{x}$$
$$+ g_8(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}, x^{(6)}, x^{(7)}) \qquad (1.2)$$

wherein a_2, a_3, a_4, a_5 are constants and g_6, g_7, g_8 and p are continuous functions depending only on the arguments shown. The function p is assumed to be ω-periodic in t, that is

$$p(t, x_1, \ldots, x_8) = p(t + \omega, x_1, \ldots, x_8)$$

for some $\omega > 0$ and for arbitrary $x_1, \ldots, x_8$.

Firstly, let us consider the eighth order constant–coefficient differential equation

$$x^{(8)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 \dddot{x} + a_6 \ddot{x} + a_7 \dot{x} + a_8 x = 0, \quad (1.3)$$

and the corresponding characteristic equation

$$\psi(r) \not\equiv r^8 + a_2 r^6 + a_3 r^5 + a_4 r^4 + a_5 r^3 + a_6 r^2 + a_7 r + a_8 = 0. \quad (1.4)$$

If β is an arbitrary real number, then

$$\psi(i\beta) = \beta^8 - a_2\beta^6 + a_4\beta^4 - a_6\beta^2 + a_8 + i\beta(a_3\beta^4 - a_5\beta^2 + a_7)$$

and if $a_3 \neq 0$, then

$$a_3\beta^4 - a_5\beta^2 + a_7 = a_3(\beta^2 - \tfrac{1}{2} a_3^{-1} a_5)^2 + a_7 - \tfrac{1}{4} a_3^{-1} a_5^2.$$

Therefore, if

$$a_3 \neq 0, \ a_8 \neq 0 \text{ and } (a_7 - \tfrac{1}{4} a_3^{-1} a_5^2) \operatorname{sgn} a_3 > 0, \quad (1.5)$$

then (1.4) cannot have any purely imaginary root. By the general theory, this, in turn, implies firstly that (1.3) has no periodic solution other than $x = 0$, and secondly that the perturbed equation

$$x^{(8)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 \dddot{x} + a_6 \ddot{x} + a_7 \dot{x} + a_8 x = p(t),$$

in which p is a continuous ω–periodic function of t, has an ω–periodic solution subject to (1.5).

The main aim of the present paper is to generalise the above ˢesults by taking into account equations (1.1) and (1.2). With reference to this we state and prove two theorems in the following part of this paper.

Theorem 1. If $a_3 \neq 0$ and

(i) $f_8(0) = 0$ and $f_8(x) \neq 0$ for $x \neq 0$,

(ii) $f_7(x_1, \ldots, x_8) \operatorname{sgn} a_3 > \tfrac{1}{4} |a_3|^{-1} f^2_5(x_1, \ldots, x_8)$

for arbitrary $x_1, \ldots, x_8$, then the equation (1.1) has no periodic solution other than $x = 0$.

Theorem 2. $a_3 \neq 0$ and assume that

(i) $\inf\limits_{x_2, x_3} g_7(x_2, x_3) \operatorname{sgn} a_3 > \tfrac{1}{4} |a_3|^{-1} a_5^2.$

(ii) $g_8(x)$ satisfies

$$g_8(x)\ \mathrm{sgn}x \to \infty \text{ as } |x| \to \infty \tag{1.6}$$

or

$$g_8(x)\ \mathrm{sgn}x \to -\infty \text{ as } |x| \to \infty; \tag{1.7}$$

(iii) there is a finite constant k such that

$$|p(t, x_1, \ldots, x_8)| \leq k$$

for all $t, x_1, \ldots, x_8$.

Then the equation (1.2) has at least one ω–periodic solution.

2. Proof of Theorem 1.

Using $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \dddot{x}$, $x_5 = x^{(4)}$, $x_6 = x^{(5)}$, $x_7 = x^{(6)}$, $x_8 = x^{(7)}$ the differential equation (1.1) can be transformed to the equivalent system

$$\begin{cases} \dot{x}_i = x_{i+1} \quad (i = 1, \ldots, 7) \\ \dot{x}_8 = -a_2 x_7 - a_3 x_6 - a_4 x_5 - f_5(x_1, \ldots, x_8)\, x_4 - f_6(x_2)\, x_3 \\ \qquad\qquad -f_7(x_1, \ldots, x_8)\, x_2 - f_8(x_1). \end{cases} \tag{2.1}$$

Let $(y_1, \ldots, y_8) \equiv (y_1(t), \ldots, y_8(t))$ be an arbitrary α–periodic solution of (2.1), that is

$$(y_1(t), \ldots, y_8(t)) = (y_1(t+\alpha), \ldots, y_8(t+\alpha)) \tag{2.2}$$

for some $\alpha > 0$.

For the completion of the proof of Theorem 1, it is enough to establish that

$$0 = y_1 = y_2 = \ldots = y_8.$$

The main tool here is the function $V(x_1, \ldots, x_8)$ which is defined by $V = U\,(\mathrm{sgn}a_3)$ where

$$\begin{aligned} U = &-x_2(x_8 + a_2 x_6 + a_3 x_5 + a_4 x_4) - \int_0^{x_1} f_8(s)\, ds - \int_0^{x_2} \eta f_6(\eta)\, d\eta \\ &- \tfrac{1}{2}(a_4 x_3^2 + a_2 x_4^2 - x_5^2) \\ &+ x_3(x_7 + a_2 x_5 + a_3 x_4 + a_4 x_3) - x_4 x_6. \end{aligned} \tag{2.3}$$

Consider the function

$$\Theta(t) \equiv V(y_1(t), \ldots, y_8(t)).$$

This function is bounded, since V is continuous and $y_1, \ldots, y_8$ are periodic in t.

A straightforward calculation gives

$$\begin{aligned}
\dot{\Theta} &= \frac{d}{dt} V(y_1, \ldots, y_8) \\
&= \{-y_3 (y_8 + a_2y_6 + a_3y_5 + a_4y_4) - y_2 [-a_2y_7 - a_3y_6 - a_4y_5 \\
&\quad -f_5 (y_1, \ldots, y_8) y_4 - f_6 (y_2) y_3 - f_7 (y_1, \ldots, y_8) y_2 - f_8 (y_1)] \\
&\quad -y_2 (a_2y_7 + a_3y_6 + a_4y_5) - y_2f_8 (y_1) - y_2y_3f_6 (y_2) \\
&\quad -a_4y_3y_4 - a_2y_4y_5 + y_5y_6 + y_4 (y_7 + a_2y_5 + a_3y_4 + a_4y_3) \\
&\quad + y_3 (y_8 + a_2y_6 + a_3y_5 + a_4y_4) - y_5y_6 - y_4y_7\} \operatorname{sgn} a_3 \\
&= \{a_3y^2_4 + f_5 (y_1, \ldots, y_8) y_2y_4 + f_7 (y_1, \ldots, y_8) y^2_2\} \operatorname{sgn} a_3 \\
&= |a_3| \{y^2_4 + |a_3|^{-1} f_5 (y_1, \ldots, y_8) y_2y_4 \operatorname{sgn} a_3\} + \\
&\quad f_7 (y_1, \ldots, y_8) y^2_2 \operatorname{sgn} a_3 \\
&= |a_3| \{y_4 + \tfrac{1}{2} |a_3|^{-1} f_5 (y_1, \ldots, y_8) y_2 \operatorname{sgn} a_3\}^2 \\
&\quad + \{f_7 (y_1, \ldots, y_8) \operatorname{sgn} a_3 - \tfrac{1}{4} |a_3|^{-1} f^2_5 (y_1, \ldots, y_8)\} y^2_2. \qquad (2.4)
\end{aligned}$$

By hypothesis (ii),

$$\dot{\Theta} \geq 0 \text{ for all } t.$$

Thus, $\Theta(t)$ is monotone in t and, recalling that $\Theta(t)$ is bounded, we have that $\lim_{t \to \infty} \Theta(t) = \Theta_0$ (constant). In view of (2.2) we can write

$$\Theta(t) = \Theta(t + m\alpha) \qquad (2.5)$$

for any arbitrarily fixed t and for arbitrary integer m, and then letting $m \to \infty$ in the right–hand side of (2.5) it follows that

$$\Theta(t) \equiv \Theta_0 \text{ for all } t. \qquad (2.6)$$

The result (2.6) implies that

$$\dot{\Theta}(t) = 0 \text{ for all } t.$$

Therefore, from (2.4) and according to hypothesis (ii) of Theorem 1, it must be that

$$y_2 = 0 \text{ for all } t. \qquad (2.7)$$

Since $\dot{y}_i = y_{i+1}$ $(i = 1, \ldots, 7)$, (2.7) in turn implies that

$$y_1 = x_0 \text{ (constant)}, \; y_3 = 0 = y_4 = y_5 = y_6 = y_7 = y_8 = \dot{y}_8 \text{ for all } t. \tag{2.8}$$

Since $(y_1, \ldots, y_8)$ is a solution of (2.1), it is evident from (2.7) and (2.8) that $f_8(x_0) = 0$; so that $x_0 = 0$, by hypothesis (i). Thus

$$(y_1, \ldots, y_8) = (0, \ldots, 0).$$

This completes the proof of Theorem 1.

3. PRELIMINARIES FOR THE PROOF OF THEOREM 2

First, it is required here to reproduce a theorem [3] which was proved by the Leray–Schauder fixed point technique. It is as follows:

Theorem. The vector differential equation

$$x' = f(t, x) \quad [f(t + \omega, x) \equiv f(t, x) \text{ continuous}]$$

admits at least one ω–periodic solution if there is a decomposition of the right–hand side

$$f = A(t)\, x + \sum_{j=1}^{n} g_j(t, x)$$

$[A(t + \omega) \equiv A(t),\; g_j(t + \omega, x) \equiv g_j(t, x)$ continuous$]$
with the following properties:

a) the homogeneous linear equation

$$x' = A(t)\, x$$

has no non–trivial ω–periodic solution;

b) there exists an a priori estimate independent of μ for the ω–periodic solutions of the equation

$$x' = A(t)\, x + \sum_{j=1}^{n} \mu^j g_j(t, x), \; 0 \leq \mu \leq 1.$$

According to this theorem, our equation (1.2) is embedded in the parameter (μ)–dependent equation

$$x^{(8)} + a_2 x^{(6)} + a_3 x^{(5)} + a_4 x^{(4)} + a_5 \dddot{x} + \mu g_6(x) \ddot{x} + g_7^*(x, \dot{x}) \dot{x} + g^*_8(x) = \mu p(t, x, \ldots, x^{(7)}) \quad (0 \leq \mu \leq 1), \tag{3.1}$$

which reduces to (1.2) when $\mu = 1$. The functions g^*_7, g^*_8 are defined by

$$g^*_7 = (1-\mu)\, b_7 \operatorname{sgn} a_3 + \mu g_7 (\dot{x}, \ddot{x}), \tag{3.2}$$

$$g^*_8 = (1-\mu)\, b_8\, x + \mu_2(x), \tag{3.3}$$

where b_7 is a constant such that

$$g_7 (x_2, x_3) \operatorname{sgn} a_3 \geq b_7 > \tfrac{1}{4} \mid a_3 \mid^{-1} a^2_5 \text{ for all } x_2, x_3 \text{ and} \tag{3.4}$$

the constant b_8 in (3.3) is fixed, positive or negative according as g_8 is subject to (1.6) or (1.7).

Also, using $x_1 = x$, $\dot{x}_i = x_{i+1}$ $(i = 1, \ldots, 7)$, the parameter (μ)–dependent equation (3.1) can be written in the system form

$$\dot{X} = AX + \mu G (X, t) \tag{3.5}$$

where

$$X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T, \; G = (0, 0, 0, 0, 0, 0, 0, \Phi)^T,$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -b_8 & -b_7 \operatorname{sgn} a_3 & 0 & -a_5 & -a_4 & -a_3 & -a_2 & 0 \end{bmatrix}$$

and

$$\Phi = p(t, x_1, \ldots, x_8) - g_6(x_2)\, x_3 + b_7 x_2 \operatorname{sgn} a_3 - g_7 (x_2, x_3)\, x_2 + b_8 x_1 - g_8 (x_1). \tag{3.6}$$

Note that G (X, t) satisfies the condition $G (X, t) = G (X, t + \omega)$ since the function p is ω–periodic in t. Furthermore, when $\mu = 0$ the system (3.5) reduces to the homogeneous linear equation

$$\dot{X} = AX \tag{3.7}$$

which has no non–trivial ω–periodic solution. In fact, the eigenvalues of A are the roots of the equation

$$r^8 + a_2 r^6 + a_3 r^5 + a_4 r^4 + a_5 r^3 + (b_7 \operatorname{sgn} a_3)\, r + b_8 = 0 \qquad (3.8)$$

which can be found from equation (1.4) by substituting $a_6 = 0$, $a_7 = b_7 \operatorname{sgn} a_3$ and $a_8 = b_8$. Hence, since

$$a_3 \neq 0, \; b_8 \neq 0 \text{ and } b_7 - \tfrac{1}{4} \mid a_3 \mid^{-1} a^2_5 > 0,$$

equation (3.8) has no purely imaginary roots, and so by the general theory, the homogeneous equation (3.7) has no non–trivial ω–periodic solution. Thus the matrix $(\bar{e}^{\omega A} - I)$, I being the identity 8 x 8 matrix, is non–singular, and hence, an ω–periodic solution of (3.5) is written in the form [5]:

$$X(t) = \mu \,(\bar{e}^{\omega A} - I)^{-1} \int_t^{t+\omega} \bar{e}^{(s-t)A}\, G\,(X(s), s)\, ds. \qquad (3.9)$$

To complete the proof of Theorem 2, it is necessary to establish that

$$\max_{0 \leq t \leq \omega} (\mid x(t) \mid + \mid \dot{x}(t) \mid + \ldots + \mid x^{(7)}(t) \mid) \leq D_0 \qquad (3.10)$$

for every ω–periodic solution x(t) of (3.1), where D_0 is a constant whose magnitude is independent of μ.

Actually it will be enough for our purpose here merely to show that

$$\max_{0 \leq t \leq \omega} (\mid x(t) \mid + \mid \dot{x}(t) \mid + \mid \ddot{x}(t) \mid) \leq D_0 \qquad (3.11)$$

for every ω–periodic solution x(t) of (3.1). Indeed, if (3.11) holds then, by hypothesis (iii) of Theorem 2, the function Φ defined by (3.6) is bounded, which implies the boundedness of the right–hand side of (3.9) subject to the standard uniform norm.

Throughout what follows $D, D_1, D_2, D_3, D_4, D_5, D_6, D_7$. denote positive constants whose magnitudes depend only on $k, a_2, a_3, a_4, a_5, b_7, b_8, g_6, g_7, g_8$ but not on μ.

4. PROOF OF THEOREM 2

Let us concider equation (3.1) and assume that g_8 is subject to (1.6). According to this assumption, b_8 is positive constant. The main tool in our verification of (3.11) is a function $W(x_1,\ldots,x_8)$, defined by

$$W = H\,(\operatorname{sgn} a_3)$$

where

$$H = -x_2\,(x_8 + a_2x_6 + a_3x_5 + a_4x_4) - \int_0^{x_1} g^*_8(s)\,ds - \mu \int_0^{x_2} \eta g_6(\eta)\,d\eta$$

$$-\tfrac{1}{2}\,(a_4x^2_3 + a_2x^2_4 - x^2_5) + x_3\,(x_7 + a_2x_5 + a_3x_4 + a_4x_3) - x_4x_6.$$

Given any solution $x(t)$ of (3.1), set

$$\varnothing_0(t) \equiv W\,(x(t), \dot{x}(t), \ldots, x^{(7)}(t)).$$

By an elementary differentiation, we get

$$\dot{\varnothing}_0 = \{a_3\,\dddot{x}^2 + a_5\,\dot{x}\,\dddot{x} + g^*_7\,(\dot{x}, \ddot{x})\,\dot{x}^2 - \mu\,\dot{x}\,p\}\ \operatorname{sgn} a_3$$

$$= |a_3|\ \{\dddot{x} + \tfrac{1}{2}\,|a_3|^{-1}\,a_5\,\dot{x}\operatorname{sgn} a_3\}^2$$

$$+\ \{g^*_7 \operatorname{sgn} a_3 - \tfrac{1}{4}\,|a_3|^{-1}\,a^2_5\}\,\dot{x}^2 - \mu\,\dot{x}p\ \operatorname{sgn} a_3.$$

By (3.2) and (3.4)

$$g^*_7 \operatorname{sgn} a_3 = (1-\mu)\,b_7 + \mu g_7\,(\dot{x}, \ddot{x})\ \operatorname{sgn} a_3$$
$$\geq b_7$$

and again from (3.4),

$$g^*_7 \operatorname{sgn} a_3 - \tfrac{1}{4}\,|a_3|^{-1}\,a^2_5 \geq D_1$$

for some D_1.

By hypothesis (iii),

$$|\mu\,\dot{x}\,p \operatorname{sgn} a_3| \leq k\,|\dot{x}|.$$

Therefore

$$\dot{\varnothing}_0 \geq |a_3|\ \{\dddot{x} + \tfrac{1}{2}\,|a_3|^{-1}\,a_5\,\dot{x}\operatorname{sgn} a_3\}^2 + D_1\,\dot{x}^2 - k\,|\dot{x}|$$

$$\geq D_2\,(\dddot{x}^2 + \dot{x}^2) - D \qquad (D = k^2/4D_1) \tag{4.1}$$

for some sufficiently small D_2.

From now onwards, we assume that x(t) is an ω–periodic solution of (3.1). By integrating both sides of (4.1) with respect to t from $t = 0$ to $t = \omega$, we get

$$0 \geq D_2 \int_0^{\omega} (\dddot{x}^2 + \dot{x}^2)\ dt - D\omega$$

and thus it follows that

$$\int_0^{\omega} \dddot{x}^2\ dt \leq D_3, \quad \int_0^{\omega} \dot{x}^2\ dt \leq D_4 \tag{4.2}$$

In view of the periodicity condition $\dot{x}(0) = \dot{x}(\omega)$ we conclude that $\ddot{x}(T_1) = 0$ at some $T_1 \in (0, \omega)$. Hence

$$\ddot{x}(t) \equiv \ddot{x}(T_1) + \int_{T_1}^{t} \dddot{x}(s)\ ds = \int_{T_1}^{t} \dddot{x}(s)\ ds$$

and, therefore,

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq \int_0^{\omega} |\dddot{x}(s)|\ ds \leq \omega^{1/2} \left(\int_0^{\omega} \dddot{x}^2(s)\ ds \right)^{1/2}$$

by Schwarz's inequality.

By (4.2), we obtain

$$\max_{0 \leq t \leq \omega} |\ddot{x}(t)| \leq \omega^{1/2}\ D_3^{1/2} \equiv D_5. \tag{4.3}$$

Now, since

$$\dot{x}(t) = \dot{x}(T_2) + \int_{T_2}^{t} \ddot{x}(s)\ ds,$$

and according to the periodicity condition $(x(0) = x(\omega))$ we can choose a $T_2 \in (0, \omega)$ such that $\dot{x}(T_2) = 0$, we obtain, in view of (4.3), the following

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq D_5\, \omega. \tag{4.4}$$

Now integrating both sides of (3.1) with respect to t and using the fact that $x(t) = x(t+\omega)$ we obtain, after some computations,

$$\int_0^\omega \{(1-\mu)\, b_8 x + \mu g_8(x)\}\, dt = \int_0^\omega \mu \{p(t, x, \ldots, x^{(7)}) - g_7(\dot{x}, \ddot{x})\, \dot{x}\}\, dt. \tag{4.5}$$

Since p is bounded and $0 \leq \mu \leq 1$, it is clear from (4.3) and (4.4) that the right–hand side of (4.5) is bounded:

$$\left| \int_0^\omega \mu \{p(t, x, \ldots, x^{(7)}) - g_7(\dot{x}, \ddot{x})\, \dot{x}\}\, dt \right| \leq D_6.$$

Now according to (4.5), there exists some $T_3 \in (0, \omega)$ such that

$$|x(T_3)| \leq D_7 \tag{4.6}$$

for some sufficiently large D_7 because g_8 is subject to (1.6) and $b_8 > 0$. Thus, from the inequality (4.6) and the identity

$$x(t) = x(T_3) + \int_{T_3}^{t} \dot{x}(s)\, ds$$

combined with (4.4), we at once get the following estimate

$$\max_{0 \leq t \leq \omega} |x(t)| \leq D_7 + D_5 \omega^2. \tag{4.7}$$

Thus, when g_8 is subject to (1.6), (3.11) holds for every ω–periodic solution of (3.1).

On the other hand, if g_8 is subject to (1.7), then b_8 must be assumed as negative constant. Therefore, the estimate (4.6) can be obtained from (4.5). Moreover, the estimates (4.3) and (4.4) also follow exactly as before. Hence, we shall have (4.7) and this completes the proof.

Example. The equation

$$x^{(8)} + 2\, x^{(6)} + x^{(5)} + x^{(4)} + \dddot{x} + \left(1 - \frac{1}{2+\dot{x}^2}\right) \ddot{x} + (1 + \dot{x}^2 + \ddot{x}^2)\dot{x}$$

$$+\, x = -2\sin t + \frac{16 \cos t}{28 + x^2 + \ddot{x}^2 + x^{(4)^2} + x^{(6)^2} + 5\,(\dot{x}^2 + \dddot{x}^2 + x^{(5)^2} + x^{(7)^2})}$$

satisfies all the hypotheses of Theorem 2, and therefore, it has at least one ω–periodic solution. In fact, the function $x = \cos t$, with

2π–periodic, is a solution of this equation. Note that the function $p(t, x, \dot{x}, \ldots, x^{(7)})$ is 2π–periodic in t.

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