Commun. Fac. Sci. Univ. Ank. Series A_1
V. 38, pp 1-6 (1989)

ON THE 3-PLANE AND CONE AND SPHERES

ABDULLAH AZİZ ERGİN*-H. HİLMİ HACISALİHOĞLU**

*Kara Harp Okulu, Ankara
**Science Faculty, Ankara University, Ankara.

SUMMARY

Elkholy and Areefi showed that in a space time, the intersection of a plane, passing through the origin, with the ligt cone, given by the equation $\sum_{i=1}^{3} x_i^2 - x_4^2 = 0$, is two 2-planes perpendicular to each other. In this study, instead of Elkholy-Areefi's ligt cone in a space time by dealing with the cone given by the equation, $a'_1 x_1^2 + a'_1 x_2^2 + a'_1 x_3^2 - b x_4^2 = 0$ and showing that also it's intersection with 3-plane, passing through the origin, is two 2-planes perpendicular to one enother, the generalization of the article of Elkholy-Areefi has been obtained. Furthermore, validity is proved for the sphere given by the equation

$\sum_{i=1}^{3} x_i^2 + \alpha = x_4^2$.

I. Introduction

1.2. *Definition*

A diametral plane is known by the equation

$$l \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} \tag{1}$$

where

$$F(x,y,z)=ax+by+cz+2fyz+2gzx+2hxy+d=0 \ [1].$$

Calculating $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$, the equation of diametral plane is obtained as

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0. \tag{2}$$

ISSN 02571-081 A.Ü. Basımevi

1.2. *Definition:*

If the normal of a diametral plane is linearly dependent to the vector (l, m, n), then the diametral plane given by (2) is called perpendicular to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} .$$

If the diametral plane is perpendicular to the line

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ then the homogeneous system of linear equations,

$$\left.\begin{aligned} (a-\lambda)\, l+hm+gn &= 0 \\ hl+(b-\lambda)\, m+fn &= 0 \\ gl+fm+(c-\lambda)n &= 0 \end{aligned}\right\} \tag{3}$$

is obtained. To have non-trivial solutions, the coefficient determinant must be zero for this equation system. That is,

$$\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-h^2-g^2-f^2)-D= 0 \tag{4}$$

wheie,

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} .$$

1.3. *Definition*

The equation,

$$\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-h^2-g^2-f^2)-D= 0$$

is called the cubic discriminating of F(x, y, z) [1].

Regarding to the equation in (4), for λ there are at most three solutions. For each λ_i, $1 \leq i \leq 3$, we can find the three non-trivial solutions (l_i, m_i, n_i). So,

$$l_i x + m_i y + n_i z = 0 , \quad 1 \leq i \leq 3 ,$$

diametral planes are obtained.

II. The Main Results

Let

$A_1x_1+A_2x_2+A_3x_3+A_4x_4 = 0$

be a 3-plane passing through the origin. Getting $\frac{A_i}{A_4} = B_i$,

$1 \leq i \leq 3$, the equation of this plane becomes,

$$\sum_{i=1}^{3} B_i x_i + x_4 = 0. \tag{5}$$

On the other hand, for the cone given by

$$\sum_{i=1}^{3} a_i' x_i^2 - b x_4^2 = 0$$

substituting $\frac{a'_i}{b} = a_i$, the equation reduces to,

$$\sum_{i=1}^{3} a_i x_i^2 - x_4^2 = 0. \tag{6}$$

From (5) and (6) we have that

$$\left(\sum_{i=1}^{3} B_i^2 x_i \right)^2 = \sum_{i=1}^{3} a_i \; x_i^2$$

or

$$\Rightarrow \sum_{i=1}^{3} (B_i^2 - a_i) x_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} B_i B_j x_i x_j = 0 ,$$

If we denote $B_i^2 - a_i = C_i^2$, we have the quadric

$$F(x_1,x_2,x_3) = C_1^2x_1^2 + C_2^2x_2^2 + C_3^2x_3^2 + 2B_1B_2x_1x_2 + 2B_1B_3x_1x_3 + 2B_2B_3x_2x_3 = 0 . \tag{7}$$

The diametral plane of this quadric can be given as,

$$l \frac{\partial F}{\partial x_1} + m \frac{\partial F}{\partial x_2} + n \frac{\partial F}{\partial x_3} = 0. \tag{8}$$

Calculating $\frac{\partial F}{\partial x_i}$, $1 \leq i \leq 3$, we can have

$$(C_1^2 l + B_1B_2 m + B_1B_3 n)x_1 + (B_1B_2 l + C_2^2 m + B_2B_3 n)x_2 + (B_1B_3 l + B_2B_3 m + C_3^2 n) = 0. \qquad (9)$$

If we consider that this diametral plane is perpendicular to the line

$$\frac{x_1}{l} = \frac{x_2}{m} = \frac{x_3}{n}$$

then we have,

$$\frac{C_1^2 l + B_1B_2 m + B_1B_3 n}{l} = \frac{B_1B_2 l + C_2^2 m + B_2B_3 n}{m} = \frac{B_1B_3 l + B_2B_3 m + C_3^2 n}{n} = \lambda,$$

and therefore we can write the homogeneous system of linear equations,

$$\left.\begin{array}{l} (C_1^2 - \lambda) l + B_1B_2 m + B_1B_3 n = 0 \\ B_1B_2 l + (C_2^2 - \lambda) m + B_2B_3 n = 0 \\ B_1B_3 l + B_2B_3 m + (C_3^2 - \lambda)\, n = 0 \end{array}\right\} . \qquad (10)$$

The cubic discriminating of the equation (7) is

$$\lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) + \lambda(C_2^2C_3^2 + C_1^2C_2^2 + C_1^2C_2^2 - B_2^2B_3^2 - B_2^2B_3^2 - B_1^2B_2^2) - D = 0 \qquad (11)$$

where

$$D = \begin{vmatrix} C_1^2 & B_1B_2 & B_1B_3 \\ B_1B_2 & C_2^2 & B_2B_3 \\ B_1B_3 & B_2B_3 & C_3^2 \end{vmatrix}$$

Substituting $B_i^2 = C_i^2 + a_i$, equation (11) becomes,

$$\lambda_3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) - \lambda[(a_2 + a_3)C_1^2 + (a_1 + a_3)C_2^2 + a_1a_2 + a_1a_3 + a_2a_3 + (a_1 + a_2)C_3^2)] - [a_2a_3C_1^2 + a_1a_3C_2^2 + a_1a_2C_3^2 + 2a_1a_2a_3] = 0. \qquad (12)$$

As a special case if we take $a_1 = a_2 = a_3 = a$ in (6) the equation (12) becomes

$$\lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) - \lambda\,[2a(C_1^2 + C_2^2 + C_3^2) + 3a^2] - [a^2(C_1^2 + C_2^2 + C_3^2) + 2a^3] = 0.$$

For the sake of shortness, if we denote, $A=C_1^2+C_2^2+C_3^2$, then the cubic equation becomes,

$$\lambda^3-\lambda^2 A-\lambda[2a.A+3a^2]-[a^2 A+2a^3] = 0 \tag{13}$$

or

$$(\lambda+a)^2[\lambda-(A+2a)] = 0 .$$

It follows that

$$\lambda_1=\lambda_2=-a \ , \text{ and } \lambda_3=A+2a .$$

Using $\lambda_1=\lambda_2=-a$ in equation (10) we have

$$B_1 l_1+B_2 m_1+B_3 n_1 = 0. \tag{14}$$

And using $\lambda_3=A+2a$ in (10) we have,

$$\left.\begin{aligned} -(B_2^2+B_3^2)l_3+B_1B_2m_3+B_1B_3n_3 &= 0 \\ B_1B_2l_3-(B_1^2+B_3^2)m_3+B_2B_3n_3 &= 0 \\ B_1B_3l_3+B_2B_3m_3-(B_1^2+B_2^2)n_3 &= 0 \end{aligned}\right\} . \tag{15}$$

Dividing the first equation of (15) by B_1 and the second by B_2 then subtracting, we have

$$\frac{l_3}{B_1} = \frac{m_3}{B_2} . \tag{16}$$

Again from the second and third equations, we can have

$$\frac{m_3}{B_2} = \frac{n_3}{B_3} \tag{17}$$

and then from (16) and (17)

$$\frac{l_3}{B_1} = \frac{m_3}{B_2} = \frac{n_3}{B_3} = k, \ k \in IR . \tag{18}$$

On the other hand, from solution (10) we have the diametral planes as

$$\left.\begin{aligned} l_1x_1+m_1x_2+n_1x_3 &= 0 \\ l_3x_1+m_3x_2+n_3x_3 &= 0 \end{aligned}\right\} \tag{19}$$

and from (18), these equations reduces to

$$\left.\begin{aligned} l_1x_1+m_1x_2+n_1x_3 &= 0 \\ B_1x_1+B_2x_2+B_3x_3 &= 0 \end{aligned}\right\}$$

where

$$<(l_1,m_1,n_1), (B_1,B_2,B_3)> = l_1B_1+m_1B_2+n_1B_3 \tag{20}$$

and from (14) it vanishes. So the planes given by (19) are perpendicular to each other.

As a result, we can write the following theorem:

II.1. *Theorem :*

The intersection of the cone $\sum_{i=1}^{3} a_i' x_i^2 = bx_4^2$ and the 3–plane $\sum_{i=1}^{4} A_i x_i = 0$ is two 2–planes, perpendicular to each other, if $a'_1 = a'_2 = a'_3$.

By using the $S\alpha$ sphere $\sum_{i=1}^{3} \alpha_i^2 + \alpha = \alpha_4^2$ instead of the cone (6) all of the results are valid. So we can express the following theorem:

II.2. *Theorem :*

In (3+1) –spacetime, the intersection of the sphere $\sum_{i=1}^{3} x_i^2 + \alpha = x_4^2$ and the hyperplane $\sum_{i=1}^{4} A_i x_i = 0$ is two 2–planes, which are perpendicular to each other.

REFERENCES

[1]. ELKHOLY E.M. and AL-AREEFI, S.M., "The 3–Plane And The Light Cone". Submitted to publish (1988).

[2]. ELKHOLY, E.M., "Curvature Of Spheres In Minkowski (n+1) space". Comm. Fac. Sci. Univ. Ankara. Volume 35, 1–2, Series A. pp. 23–25 (1987).