Commun. Fac. Sci. Univ. Ank. Series A_1
V. 41. pp. 45-48 (1992)

SUBRINGS OF THE RING OF ALL ANALYTIC FUNCTIONS ON AN OPEN RIEMANN SURFACE

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(Received Nov. 29, 1991; Accepted July 14, 1992)

ABSTRACT

Let G be a region in an open Riemann surface R and A(G) be a ring (algebra) of single valued analytic functions on G. We studied some subrings of A(G) corresponding discrete sets in G and we give some algebraic characterization of them.

1. INTRODUCTION

It has been known that the structure of an open Riemann surface is determined by the algebraic structure of certain rings of analytic functions on it. Let X_i be plane domain and $A(X_i)$ be the ring of all analytic functions on X_i $(i = 1, 2)$. If there is an isomorphism from $A(X_1)$ to $A(X_2)$, we write $A(X_1) \cong A(X_2)$. X_1 and X_2 domains are said to be conformally equivalent to each other if there exists a single-valued analytic function φ which maps X_1 conformally onto X_2. If X_1 and X_2 are conformally equivalent, we write $X_1 \sim X_2$ L. Bers [2] has shown that if $A(X_1) \cong A(X_2)$, then $X_1 \cong X_2$. For arbitraty open Riemann surfaces X and Y, W. Rubin [5] and H.L. Royden [4] proved the above fact under the assumtion that the given isomorphism preserves complex constants. But M. Nakai [3] proved without such a prior assumption for complex constants. It is well known from [3] that $A(G_1) \cong A(G_2)$ iff $G_1 \sim G_2$. In the case that G_i is any non-empty subset in an open Riemann surface R_i, $i = 1.2$, A. Şerbetci and İ.K. Özkın solved the same problem [6].

2. SUBRINGS OF A(G).

Definition 1. Let R be an open Riemann surface, X be any non-empty subset of R and, f be a function from X into C, the complex plane, f is said to be analtyic on X if there is an open set $U \supset X$ and an analytic function F on U such that $F/X = f$. This definition is equivalent to requiring if there is an open neighbourhood U_P of p for every $p \in X$ and an analytic fuction F_P: $U_P \rightarrow C$ on U_P such that $F_P / (U_P \cap X) = f / (U_P \cap X)$.

Definition 2. Let R_1 and R_2 be two open Riemann surfaces and, X and Y be any non–empty subsets of R_1 and R_2 respectively. A mapping φ from X to Y is said to be analytic mapping if φ is analytic function on X and has values in Y. φ is said to be a conformal mapping and is written $X \sim Y$ if φ is analytic, one–to–one. and onto. X and Y are said to be conformally equivalent if there is a one–to conformal mapping from X onto Y.

Let R_1 and R_2 be two open Riemann surfaces and, X and Y be any nonempty subsets of R_1 and R_2 respectively. We will denote by A(X) and A(Y) the rings of all single–valued analytic functions on X and Y respectively. It is well known that $A(X) \cong A(Y)$ iff $X \sim Y$ [6].

Definition 3. Let R be a Riemann surface and E be subset of R. E is called a discrete set if there is not any accumulation point of E. Since an open Riemann surface has a countable base, a dicrete set is countable ([1], s. 144).

Theorem 1. Let G be a region in an open Riemann surface and E be a discrete subset of G. We consider the complex Banach algebra B(G) of bounded analytic functions with respect to supremum norm. Then

$$B(E) = \{f \in B(G): f(p) = \text{constant},\ p \in E\}$$

is a Banach subalgebra of B(G).

Proof: To prove the theorem it suffices to show that B(E) is closed. Let $g \in \overline{B(E)} \cap B(G)$.

Then there exists a sequence $(f_n) \subset B(E)$ such that

$$d(f_n, g) = \|f_n - g\| = \sup\{|f_n(w) - g(w)| : w \in G\} \rightarrow 0, n = 1,2,\ldots$$

Since $f_n(p) = \text{constant}$,

$$(f_n(p)) = (a_n) \rightarrow q(p).$$

, where $a_n = f_n(p)$. This shows that g is element of B(E) and B(E) is closed.

Theorem 2. Let G_1 and G_2 be two regions in an open Riemann surface and $E_1 \subset G_1$ and $E_2 \subset G_2$ be two discrete subsets. Furthermore we suppose that φ is a conformal mapping from G_2 to G_1 such that $\varphi(E_2) = E_1$ and

$$B(E_1) = \{f \in B(G_1): f(p) = \text{constant},\ p \in E_1\}$$

$$B(E_2) = \{g \in B(G_2): g(q) = \text{constant},\ q \in E_2\}.$$

Then $\Phi: B(E_1) \rightarrow B(E_2)$, $\Phi(f) = f \circ \varphi$ is a C – isomorphism.

Proof: We at once point out that a C–isomorphism is a isomorphizm that preserves complex constant. It is clear that $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$ and $\Phi(f_1 f_2) = \Phi(f_1)\,\Phi(f_2)$, that is Φ is a ring homomorphism. To show that Φ is onto let g be an arbitrary element of $B(E_2)$. Then since $\varphi^{-1}(p) \in E_2$ for $p \in E_1$, $g(\varphi^{-1}(p))$ is constant. So $g \circ \varphi^{-1} \in B(E_1)$ and

$$\Phi(g \circ \varphi^{-1}) = (g \circ \varphi^{-1} \circ \varphi) = g.$$

On the other hand, since $\Phi(f_1) = \Phi(f_2) \Rightarrow f_1 \circ \varphi = f_2 \circ \varphi \Rightarrow f_1 = f_2$, Φ is one–to– one. Furthermore since for $\alpha \in C$, $\Phi(\alpha) = \alpha$, Φ is a C–isomorphism.

Corollary 1: If $E_1 \subset E_2$ then $B(E_2) \subset B(E_1)$. In particular if $E_1 = \{p\}$ is a singleton set then $B(E_1) = B(p) = B(G_1)$.

Proof: It is clear that $B(E_2) \subset B(E_1)$. If $E_1 = \{p\}$ then since $f(p) = a \in C$ for each $f \in B(G_1)$, $B(G_1) \subset B(p)$ and hence $B(G_1) = B(E_1)$.

Corollary 2: If $E_1 \subset E_2 \subset \ldots \subset E_n$ and E_i is a discrete subset of G then $B(E_n) \subset B(E_{n-1}) \subset B(E_1), 1 \geq i \geq n$.

Let f be any element of A (G) and $Z(f)$ denote the set of zeros of f. Namely, $Z(f) = \{p \in G: f(p) = 0\}$. If $f \not\equiv 0$, then $Z(f)$ is a discrete set. The set

$$J(Z(f))) = \{f \in A(G): f(p) = 0,\ p \in Z(f)\}$$

is an ideal in A(G). If $Z(f_1) \subset Z(f_2) \subset \ldots \subset Z(f_n)$, then by the above corollary 2, we have

$$J(Z(f_n)) \subset J(Z(f_{n-1})) \subset \ldots \subset J(Z(f_1)).$$

We now can give the following theorem.

Theorem 3. Let $\Phi: A(G_1) \rightarrow A(G_2)$ be a C-isomorphism defined by $\Phi(f) = f \circ \varphi$ for all $f \in A(G_1)$. If $\varphi(f_1) = f_2$ then $\varphi(Z(f_2)) = Z(f_1)$.

Proof: It is well known that if $A(G_1) \cong A(G_2)$, then G_2 and G_1 are conformaly equivalent. Since $f_2 = f_1 \circ \varphi$ by the hypothesis for each $q \in Z(f_2)$

$$0 = f_2(q) = f_1(\varphi(q)).$$

Thus $\varphi(q) \in Z(f_1)$ and consequently

$$\varphi(Z(f_2)) = \{\varphi(q) : q \in Z(f_2)\} \subset Z(f_1).$$

Similarly, it can be shown that $Z(f_1) \subset \varphi(Z(f_2))$ and we have $Z(f_1) = Z(f_2)$.

Let

$$A(Z(f_1)) = \{f \in A(G_1) : f(p) = \text{constant},\ p \in Z(f_1)\}$$

$$A(Z(f_2)) = \{g \in A(G_2) : g(q) = \text{constant},\ q \in Z(f_2)\}$$

We can give the following theorem.

Theorem 4. Let Φ be a C–isomorphism from $AG_1)$ to $A(G_2)$ and $\Phi(f_1) = f_2$. Then $A(Z(f_1) \cong A(Z(f_2))$.

Proof: Since Φ is a C–isomorphism there exists a conformal mapping φ from G_2 to G_1. Then by the above Theorem 3 $\varphi(Z(f_2)) = Z(f_1)$. So by the Theorem 2 we have

$$A(Z(f_1)) = A(Z(f_2)).$$

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