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ON CIRCLES AND SPHERES IN GEOMETRY

MURATHAN, C., ÖZDAMAR, E.

Department of Mathematics Uludağ University Art. and Sci. Fac. 16059 Görükle/ Bursa / Türkiye.

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ABSTRACT

In [1] the main result (Teorem 2) which states that "a submanifold M^n of a RiemanSian maniforld $\tilde{M}^n$ is an extrinsic sphere (i.e. the submanifold which is umbilic and has parallel mean curvature) iff every circle in $\tilde{M}^n$ is a circle in M^m must have some additional hypothesis. We pointed out these hypothesis just the special case of $M^m = IR^m$

1. INTRODUCTION

Let i be the inclusion map of (M^n, g) into IR^n where g denotes the induced Riemannian metric on M^n from IR^n. Then the following equalities hold

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ \tilde{\nabla}_X \zeta &= - A_\zeta X + D_X \zeta \end{aligned} \qquad (1.1)$$

for every vector fields X, Y on M^n and ζ ortogonal to M, where $\tilde{\nabla}$, ∇, D denote the connections on IR^n, M^n and the normal connection on M^n respectively. h is the second fundamental form of M^n and A_ζ denotes the shape operator with respect to ζ. The second fundamental form h and the shape oper ator A_ζ have the following relation

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta) \qquad (1.2)$$

If $\tilde{\nabla}_X h \equiv O$ for every vector field X on M^n then we call M^n a parallel immersed submanifold of IR^n (i.e. the inclusion map i: $M^n \to IR^m$ parallel immersion) where $\tilde{\nabla}$ denotes the Van der-Waerden Bortolotti connection on M^n. The vector field

$$H = (1/n) \sum_{j=1}^{n} h(e_j, e_j)$$

called the mean curvature vector field where $\{e_1, \ldots, e_n\}$ is an orthonormal vector fields on M. If $D_X H = O$ for every X then we call that (M^n, g) has parallel mean curvature vector.

Secondly, we recall the notion of planar sections from [2]. $p \in M^n$ and $t \in T_P M$ where $T_P M$ denotes the tangent space to M at the point p. Affin subspace E(p, t) of IR^n is defined as the subspace that corresponds to the vector subspace $S_P \{t, TM^{\perp}\}$ at the point p expressed as

$$E(p, t) = p + S_P \{t, TM^{\perp}\}$$

where $TM^{\perp}$ denotes the orthogonal complemantary vector subspace of $T_P M$ in $T_P IR^m$ and $S_P \{t, TM^{\perp}\}$ is vector subspace spanned by t and $TM^{\perp}$.

The intersection curve $M^n \subset E(p, t)$ will be called the normal section of M^n through t at the point p and denoted by ns(M, p,t). If the curve ns(M, p, t) is an 2–dimensional plane curve in IR^m for every $t \in T_P M$ and every $p \in M$ then we call that M has 2–planar normal sections. If the curve ns (M, p, t) doesn't have to be 2–planar globally but locally, near the point p, then we call M^n that has pointwise 2–planar normal sections. In that case if a parametrization of ns (M, p, t) will be given as $s \to \gamma(s)$ then

$$\gamma'(0) \wedge \gamma''(0) \wedge \gamma'''(0) = 0; \; (\gamma(0) \quad = p)$$

vice verse [2], where Λ denotes the exterior product of the vectors $\gamma'(0)$, $\gamma''(0)$, $\gamma'''(0)$.

It is clear that the submanifold M^n of IR^m has 2–planar normal sections then it has pointwise 2–planar normal sections.

We recall three main results from [2] about pointwise 2–planar-sections and a general result of [3] as follows:

Theorem. A: If M^n has pointwise 2–planar normal sections iff

$$(\tilde{\nabla}_t h)(t, t) \wedge h(t, t) = 0 \qquad (1.3)$$

for every $t \in T_P M^n$ [2].

Theorem. B: If M^n has pointwise 2–planar normal sections the point p is a vertex of the normal section curves then

$$(\tilde{\nabla}_t h)(t, t) = 0 \qquad (1.4)$$

for every $t \in T_P M^n$ and vice versa [2].

Theorem. C: M^n has planar geodesics iff it has planar normal sections of the same constant curvature [2].

Theorem. D: Let f: $M^n_r \to IR^N_a$ be an isometric immersion of connected pseudo-Riemannian manifold, $n \geq 2$. If every non-null gedesics c of M, foc is a plane curve in IR^N_a, then $L = g(h(X, X), h(X, X))$ is constant for all unit vectors $X \in TM$, and we have the following cases:

$L > 0$: Each foc is part of $S^1_1 \subset IR^2_1$ or an $S^1 \subset IR^2$, O each of radius $(1 / \sqrt{L})$.

$L < 0$: Each foc is part of an $IH^1 \subset IR^2_2$ or an $IH^1_1 \subset IR^2_2$, each of radius $(-1 / \sqrt{L})$.

$L = 0$: Each foc is either a line segment or a curve in a degenerate plane $IR^2_{0,1}$, or $IR^2_{1,1}$ [3].

Finally, a circle $s \to \gamma(s)$ satisfies the following differential equation of third order

$$\nabla_X \nabla_X X + g(\nabla_X X \nabla_X X)\, X = 0 \tag{1.5}$$

where $\gamma'(s) = X(\gamma(s))$, [1].

2. A CHARACTERIZATION FOR SPHERES IN IR^m

In this section, we shall prove

Theorem 1: Let M^n be an n-dimensional connected Riemannian submanifold of IR^m. If M^n is total umbilic and has parallel mean curvature vector then M^n has pointwise 2-planar normal sections and for some $k > 0$, every circle of radius k in M^n is a circle in IR^m.
And conversly;

Theorem 2: Let M^n be an n-diemensional connected Riemanian submanifold of IR^m. If

i) every circle in M^n is a circle inIR^m

ii) M^n has planar geodesics

iii) dimension of $(im(h)) = 1$ at every point of M^n where
$im(h) = \{h(t, t) \mid t \in T_PM\}$

and h is the second fundamental form of M^n

then M^n is totally umbilic and has parallel mean curvature vector.

Remark: In [1], Theorem. 2 states nearly same thing as the above two theorems. The proof which is given in [1] and based on" changing Y into -Y" where Y is the second Frenet vector field and defined as the following:

$$(\nabla_{X_s} X_s)_{s=0} = (1/r)\, Y$$

and the vector fields system $\{X, Y\}$ satifies the following equations (in [1], eq. (1), (2))

$$\left.\begin{aligned} \nabla_{X_s} X_s &= k\, Y_s \\ \nabla_{X_s} Y_s &= -k\, X_s \end{aligned}\right\} \qquad (2.1)$$

But the system $\{X, -Y\}$ doesn't satisfies the equation (2.1). In fact, if we assume that

$$\nabla_{X_s} X_s = k\, Y$$

$$\nabla_{X_s} Y_s = -k\, X_s,\ k > 0$$

then $k(-Y_s) = -\nabla_{X_s} X_s \Rightarrow \nabla_{X_s} X_s = kY_s$

but $\nabla_{X_s}(-Y_s) = -\nabla_{X_s} Y_s = -(-kX_s) = kX_s$

so $\nabla_{X_s}(-Y_s) \neq -k\, X_s$

thus the equations false for $\{X, -Y\}$.

Proof of Theorem 1: Let p be an arbitrary point of M and t, y are orthonormal vectors in T_pM. If γ is a circle in M such that

$$\gamma(0) = p$$
$$\gamma'(0) = t$$

that Frenet equations of γ as the following:

$$\left.\begin{aligned} (\nabla_X X)(0) &= \nabla_t X = y \\ (\nabla_X Y)(0) &= \nabla_t Y = -k\, t \end{aligned}\right\} \qquad (2.2)$$

where $\{X, Y\}$ uniquely determined by the equations (2.1) with initial conditions

$$X(0) = t$$
$$Y(0) = y$$

Since γ is a circle, then

$$\nabla_X \nabla_X X + g(\nabla_X X, \nabla_X X) X = 0. \tag{2.3}$$

M is totally umbilic, that is,

$$h(W, Z) = g(W, Z) H, \text{ for every } W, Z \in TM$$

then

$$h(\nabla_X X, X) = g(\nabla_X X, X) H = k\, g(Y, X) H \tag{2.4}$$

and

$$A_{h(x,x)} X = g(H, H) X. \tag{2.5}$$

Now, by the equations (1.1) we have

$$\tilde{\nabla}_X \tilde{\nabla}_X X = \nabla_X \nabla_X X + h(X, \nabla_X) - A_{h(X,X)} X + D_X h(X, X) \tag{2.6}$$

Substituting (2.4) and (2.5) into (2.6) we obtain

$$\tilde{\nabla}_X \tilde{\nabla}_X X = \nabla_X \nabla_X X - g(H, H) X \tag{2.7}$$

Secondly, we calculate $g(\tilde{\nabla}_X X, \tilde{\nabla}_X X)$ by using the equations (1.1) together with (1.2) and bilinearity of g as follows

$$\begin{aligned} g(\tilde{\nabla}_X X, \tilde{\nabla}_X X) &= g(\nabla_X X + h(X, X), \nabla_X X + h(X, X)) \\ &= g(\nabla_X X, \nabla_X X) + g(H, H) \end{aligned} \tag{2.8}$$

Thus, by (2.3), (2.7) and (2.8) we obtain

$$\begin{aligned} \tilde{\nabla}_X \tilde{\nabla}_X X + g(\tilde{\nabla}_X X, \tilde{\nabla}_X X) X &= \nabla_X \nabla_X X + g(H, H) X + \\ & g(\nabla_X X, \nabla_X X) X - g(H, H) X \\ &= 0. \end{aligned}$$

That is, γ is a circle in IR^m

On the other hand M^n has parallel mean curvature vector so

$$D_X h(X, X) = D_X H = 0.$$

Thus

$$(\nabla_t h)(X, X) = D_t h(X, X) - 2h(t, \nabla_t X) = 0$$

and a consequence of that;

$$(\nabla_t h)(t, t) \wedge h(t, t) = 0$$

which shows by Theorem. A that M^n has pointwise 2–planar normal sections. O.E.D

Proof of Theorem. 2: By the Theorem. C M^n has 2–planar normal sections of the same constant curvature. Thus M^n satisfies the hypothesis of Theorem. D so the geodesic γ' with the initial condition $x \in$ TM is an arc on a circle S^1 of costant radius

$$\frac{1}{\| h(x, x) \|}$$

passes through X in TM. This arch by [3], is the solution curve of the following initial–value problem:

$$\left.\begin{aligned} \tilde{\nabla}_X X &= \| h(X, X) \| Y \\ \tilde{\nabla}_X Y &= -\| h(X, X) \| X \\ X(p) &= x, \; Y(p) = y \end{aligned}\right\} \tag{2.9}$$

Furthermore,

$$Y = \frac{h(x, x)}{\| h(x, x) \|}$$

thus Y is unique up to $\pi(x)$ and independent of choice of x, at the point $\pi(X)$ since dim (imh) = 1 so the curvature centre of γ is the point c satifies the following equation

$$c = \pi(x) + \frac{1}{\| h(x, x) \|} Y_{\pi(X)}$$

which is independent of the choice of x, that is c is constant what we have shown that every point of M^n has a hyperspherical neighbor hood in IR^m which can be considered as neighborhood of that point in M^n. Thus M^n is totally umbilic and has parallel mean curvature vector.

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