CONVEX AND STARSHAPED SUBSETS IN MANIFOLDS PRODUCT

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ABSTRACT

Convexity and starshape concepts in the cartesian product of two complete simply connected smooth Riemannian manifolds without conjugate points are studied in terms of the same concepts in the components of the product.

1. INTRODUCTION

In [10], Pandey introduced an interesting form of a Riemannian metric g and connection D on the Cartesian product $M_1 x M_2$ of two C^∞ Riemannian manifolds M_1 and M_2. Through studying the properties of g and D in addition to other geometric characteristics, Pandey proved some interesting results concerning the product of almost complex manifolds, Kahlerian manifolds as well as almost Tachibana manifolds. Other properties of the Ricci tensor of the manifolds product have also been established in [10].

Utilizing the study of [10], we established some geometric results concerning conjugate as well as focal points in the Cartesian product $M_1 x M_2$ of Riemannian manifolds [2]. Among the results of [2] we proved that the product $M_1 x M_2$ of two C^∞ Riemannian manifolds is free from conjugate (resp. focal) points under the metric and connection given in [10] if and only if both M_1 and M_2 are free from conjugate (resp. focal) points under their own metrics and connections. For the detailed study see [2].

The main goal of the present work is to use the principal results of both [2] and [10] to study the convexity and starshape concepts in the Cartesian product of Riemannian manifolds without conjugate points.

In the following, we suppose that the reader has some familiarity with [2, 3, 4, 5, 8, 9] for main facts related to conjugate as well as focal points. From now on, all manifolds, maps, vector fields, ... etc. are assumed sufficiently smooth for computations to make sense.

2. ON MANIFOLDS CARTESIAN PRODUCT

Let M_1 and M_2 be two C^∞ complete Riemannian manifolds with Riemannian metrics g_1 and g_2 and Riemannian connections $D^{(1)}$ and $D^{(2)}$, respectively. A Riemannian metric g on $M_1 \times M_2$ may be defined as follows [10]

$$\begin{aligned} g\,(X,\,Y) &= g\,((X_1,\,X_2),\,(Y_1,\,Y_2)) \\ &= g_1\,(X_1,\,Y_1) + g_2\,(X_2,\,Y_2) \end{aligned} \tag{2.1}$$

where $X_i,\,Y_i \in \mathcal{H}\,(M_i)$ and $\mathcal{H}(M_i)$ denotes the set of all vector fields on M_i, $i = 1, 2$. Similarly, a Riemannian connection D on $M_1 \times M_2$ may be given by [10]

$$D_X Y = D_{(X_1,\,X_2)}\,(Y_1,\,Y_2) = (D_X^{(1)}\,Y_1,\,D_X^{(2)}\,Y_2) \tag{2.2}$$

It is not difficult to prove that D is metric as well as torsion free connection. For details see [2, 10].

A nice example of the Cartesian product of two Riemannian manifolds under g and D mentioned above is the Euclidean space $(E^2,\,<,\,>)$ where $<,\,>$ denotes the usual metric on E^2. We can write $E^2 = E^1 \times E^1$. If we consider $X,\,Y \in \mathcal{H}\,(E^2)$, we have that

$$\begin{aligned} <X,\,Y> &= <(X_1,\,X_2),\,(Y_1,\,Y_2)> = X_1 Y_1 + X_2 Y_2 \\ &= <X_1,\,Y_1> + <X_2,\,Y_2> \end{aligned}$$

Taking into account that the connection in this case is the usual differentiation when acting on real functions, we have in terms of the Cartesian coordinates

$$D_{(x_1,\,x_2)}\,(Y_1,\,Y_2) = D_{x_1\frac{\partial}{\partial x}\,+\,x_2\frac{\partial}{\partial y}}\,(Y_1,\,Y_2)$$

$$= (D_{x_1\frac{\partial}{\partial x}\,+\,x_2\frac{\partial}{\partial y}}\,Y_1,\,D_{x_1\frac{\partial}{\partial x}\,+\,x_2\frac{\partial}{\partial y}}\,Y_2)$$

$$= (D_{X_1}\frac{\partial}{\partial x} Y_1, D_{X_2}\frac{\partial}{\partial y} Y_2)$$

$$= (D_{X^{(1)}_1} Y_1, D_{X^{(2)}_2} Y_2).$$

With respect to the above mentioned g and D we can easily show that tangents to one manifold are regarded as being perpendicular to those of the other, i.e for $X_1 \in \mathscr{H} (M_1)$ and $X_2 \in \mathscr{H} (M_2)$, we have that $(X_1, 0), (0, X_2) \in \mathscr{H} (M_1 \times M_2)$ are orthogonal since

$$g ((X_1, 0), (0, X_2)) = g_1 (X, 0) + g_2 (0, X_2) = 0$$

If $\gamma: [0, \lambda] \to M_1 \times M_2$ is a smooth curve in $M_1 \times M_2$, then the natural projections $\gamma_1: [0, \lambda] \to M_1$ and $\gamma_2: [0, \lambda] \to M_2$ of γ on both M_1 and M_2, respectively, are smooth curves. Moreover, γ is a geodesic in $M_1 \times M_2$ if and only if both γ_1 and γ_2 are geodesics in M_1 and M_2, respectively. This claim may be verified as follows:

$$D_{\dot{\gamma}} \dot{\gamma} = D_{(\dot{\gamma}_1, \dot{\gamma}_2)} (\dot{\gamma}_1, \dot{\gamma}_2) = (D^{(1)}_{\dot{\gamma}_1} \dot{\gamma}_1, D^{(2)}_{\dot{\gamma}_2} \dot{\gamma}_2) \qquad (2.3)$$

where $\dot{\gamma}$ is the velocity vector field along the curve γ. Consequently, $D_{\dot{\gamma}} \dot{\gamma} = 0$ if and only if $D^{(i)}_{\dot{\gamma}_i} \dot{\gamma}_i = 0$ for $i = 1, 2$.

Let $\gamma_1: [0, \lambda] \to M_1$ and $\gamma_2: [0, \mu] \to M_2$ be two smooth curves in M_1 and M_2, respectively. Through a simple linear transformation between $[0, \lambda]$ and $[0, \mu]$ we can reparametrize the curve γ_2, say, on the interval $[0, \lambda]$ and consequently one can talk about segment curves in W_1 and W_2 as being defined on the same interval.

3. ON CONVEX AND STARSHAPED SUBSETS

Definition (3.1)

A subset B of a Riemannian manifold M is convex if for each pair of points. a, b $\in$ B there exists a unique minimal geodesic segment γ joining a and b in M such that $\gamma \subset$ B. A convex subset B $\subset$ M will be called convex body if B has non–empty interior [11].

Definition (3.2)

For a subset B $\subset$ M, the convex hull of B is defined to be the smallest convex subset of M containing B and denoted by CH(B).

Definition (3.3)

A subset B of a Riemannian manifold M is starshaped (starlike) with respect to the point $p \in B$ if for each point $q \in B$ there exists a unique minimal geodesic segment γ in M joining p and q such that $\gamma \subset B$. We say that p sees all the points of B via B [6, 7] or all points of B are visible from p via B. The collection of such a point $p \in B$ is called the kernel of B and denoted by ker B.

It is easy to show that for a starshaped subset $B \subset M$, ker B is itself a convex subset of M. Moreover, if B is a subset of a Riemannian manifold M, then B is convex of and only if $B = \ker B$. These two statements may be considered as a generalization of a result proved by H. Brunn (See [11] p. 5).

If M is a complete simply connected C^∞ Riemannian manifold without conjugate points one can omit completely the word "unique minimal" in the above definitions $(3.1) - (3.3)$ as in such a type of manifolds each pair points has a unique connecting geodesic segment [4, 8, 9].

From now on let us take W_1 and W_2 to be complete simply connected C^∞ Riemannian manifolds without conjugate points. Using proposition (1–3) [2] we have that $W_1 x W_2$[*] is also a complete simply connected C^∞ Riemannian manifold without conjugate points. Notice that dim $(W_1 x W_2) =$ dim $(W_1) +$ dim (W_2). Consequently, each pair of different points (p_1, p_2) and (q_1, q_2) in $W_1 \times W_2$ are joined by a unique geodesic segment γ. This segment when naturally projected on W_1 and W_2 yields two geodesic segments $\gamma_i \subset W_i$ joining p_i and q_i, $i = 1, 2$ each one is unique in its own manifold (Section 2). The natural projection will be denoted by

$$\pi_i: W_1 \times W_2 \to W_i$$

where $\pi_i (p_1, p_2) = p_i$, $i = 1, 2$.

Proposition (3.4)

Let $B_1 \subset W_1$ and $B_2 \subset W_2$ be subsets of W_1 and W_2. Then $B_1 \times B_2 \subset W_1 \times W_2$ is convex if and only if both B_1 and B_2 are convex.

Proof:

Firstly, assume that $B_1 \times B_2$ is a convex subset of $W_1 \times W_2$. Consider $p_1, q_1 \in B_1$ and $p_2, q_2 \in B_2$ to be arbitrary points. Let

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*The metric and connection on $W_1 x W_2$ are as given in Section 2.

$\gamma_i: [0, \lambda] \to W_i$ be the geodesic segment joining p_i and q_i, $i = 1, 2$. Consider the curve $\gamma: [0, \lambda] \to W_1 \times W_2$ defined by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $t \in [0, \lambda]$. As we mentioned in Section 2, it is clear that γ is a geodesic segment joining the points (p_1, p_2), $(q_1, q_2) \in B_1 \times B_2$. By convexity of $B_1 \times B_2$ we have $\gamma \subset B_1 \times B_2$. If we project the segment γ naturally on W_1 and W_2, we have that $\pi_i\gamma = \gamma_i \subset B_i$, $i = 1, 2$, which shows that both B_1 and B_2 are convex subsets of W_1 and W_2, respectively.

For sufficiency, assume that both $B_1 \subset W_1$ and $B_2 \subset W_2$ are convex subsets. Consider any arbitrary pair of points (p_1, p_2) and (q_1, q_2) in $B_1 \times B_2 \subset W_1 \times W_2$. Let γ be the geodesic segment in $W_1 \times W_2$ joining (p_1, p_2) and (q_1, q_2). We show that $\gamma \subset B_1 \times B_2$.

Assume in contrary that $\gamma \not\subset B_1 \times B_2$. Consequently, there exists a point $(c_1, c_2) \in \gamma$ such that $(c_1, c_2) \notin B_1 \times B_2$. In this way, we have that one —at least— of the following statements $c_1 \notin B_1$, $c_2 \notin B_2$ is satisfied. Without loss of generality, assume that $c_1 \notin B_1$. Considering the natural projection $\pi_1\gamma$ of γ, we obtain a geodesic segment $\gamma_1 = \pi_1\gamma$ joining p_1 and q_1 in W_1 such that $c_1 \in \gamma_1$ and $c_1 \notin B_1$. Consequently, B_1 would be a non–convex set in W_1 contradicting the assumption. Same discussion can be carried out in the cases $c_2 \notin B_2$ or $c_1 \notin B_1$, $c_2 \notin B_2$ and the proof of the sufficiency part is now complete.

In the light of Proposition (3.4) we can prove the following.

Corollary (3.5)

Let $B \subset W_1 \times W_2$ be a convex subset. Then the natural projections $B_i = \pi_i B \subset W_i$ of B onto W_i, $i = 1, 2$, are both convex subsets.

It is worth mentioning that the converse of Corollary (3.5) is not necessarily true, i.e. for a sunset $B \subset W_1 \times W_2$, the natural projections $\pi_i B \subset W_i$, $i = 1, 2$, might be convex although B is itself non–convex. Fig. (1) shows this fact. Notice that in this case,

$$\pi_1 B \times \pi_2 B \neq B.$$

The following example may be considered as an application of Proposition (3.4).

Example (3.6)

Consider the two convex subsets $A \subset E^1$ and $B \subset E^2$ where $A = [0, 1]$ and $B = \{(x, y): x^2 + y^2 \leq 1\}$ the unit disc. The subset
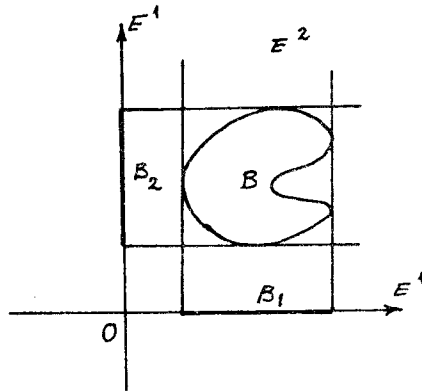
Fig. (1)

A x B ⊂ E¹ x E² = E³ which represents a truncated cylinder is clearly a convex subset of E³. (See Fig. (2)).

Proposition (3.7)

Let $B_1 \subset W_1$ and $B_2 \subset W_2$ be two subsets of W_1 and W_2. Then
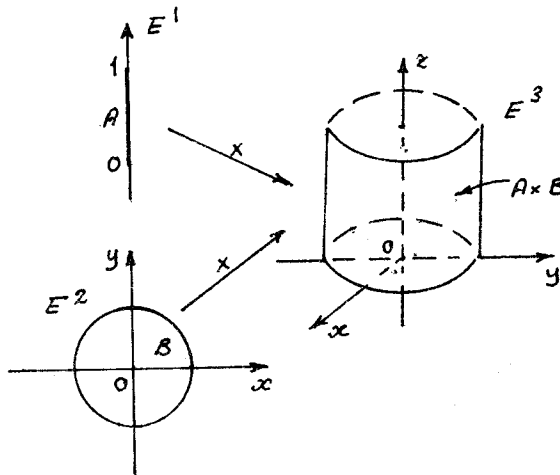
$$CH \ (B_1 \ x \ B_2) \ = \ CH \ (B_1) \ x \ CH \ (B_2).$$

Fig. (2)

Proof:

Let us assume that $CH(B_1 \times B_2)$ is a proper subset of $CH(B_1) \times CH(B_2)$. Consequently, $B_1 \times B_2 \subset CH(B_1 \times B_2) \subset CH(B_1) \times CH(B_2)$. If we naturally project $CH(B_1 \times B_2)$ as a convex subset of $W_1 \times W_2$ onto W_1 and W_2, then one of the two projections, say $\pi_1 CH(B_1 \times B_2)$, will be a convex proper subset of $CH(B_1)$ containing B_1 which contradicts the assumption that $CH(B_1)$ is the convex hull of B_1.

Finally, if we assume that $CH(B_1) \times CH(B_2)$ is a proper subset of $CH(B_1 \times B_2)$, then as $CH(B_1) \times CH(B_2)$ (Proposition (3.4) is a convex subset of $W_1 \times W_2$ containing $B_1 \times B_2$, $CH(B_1 \times B_2)$ will not be the convex hull of $B_1 \times B_2$ which is again a contradiction.

From the above discussion we have that $CH(B_1 \times B_2) = CH(B_1) \times CH(B_2)$ and the proof is complete.

Proposition (3.8)

Let $B_1 \subset W_1$ and $B_2 \subset W_2$ be two subsets. Then

(i) $B_1 \times B_2 \subset W_1 \times W_2$ is starshaped if and only if both B_1 and B_2 are starshaped.

(ii) $\ker (B_1 \times B_2) = (\ker B_1) \times (\ker B_2)$

Proof of Part (i)

Firstly, assume that $B_1 \times B_2 \subset W_1 \times W_2$ is starshaped with respect to the point $(p_1, p_2) \in B_1 \times B_2$. We prove that B_i is starshaped with respect to $p_i \in B_i$, $i = 1, 2$.

Now, we show that B_1 is starshaped with respect to p_1. Consider any arbitrary point $q_1 \in B_1$. Since $B_1 \times B_2$ is starshaped with respect to (p_1, p_2) so there exists a geodesic segment $\gamma \colon [0, \lambda] \to W_1 \times W_2$ such that $\gamma(0) = (p_1, p_2), \gamma(\lambda) = (q_1, q_2)$ and $\gamma \subset B_1 \times B_2$. Projecting γ onto W_1 we obtain that $\pi_1 \gamma = \gamma_1$ is a geodesic segment contained in B_1 and so B_1 is starshaped with respect to p_1.

Similar argument shows that B_2 is starshaped with respect to p_2.

Conversely, let $B_1 \subset W_1$ and $B_2 \subset W_2$ be starshaped subsets with respect to the points $p_1 \in B_1$ and $p_2 \in B_2$, respectively. We shall prove that $B_1 \times B_2 \subset W_1 \times W_2$ is starshaped with respect to the point $(p_1, p_2) \in B_1 \times B_2$.

Consider any arbitrary point $(q_1, q_2) \subset B_1 \times B_2$ and let $\gamma \colon [0, \lambda] \to W_1 \times W_2$ be the unique geodesic segment joining (p_1, p_2) and (q_1, q_2) such that $\gamma(0) = (p_1, p_2)$, $\gamma(\lambda) = (q_1, q_2)$. If we project γ onto W_1 and W_2 we obtain a unique geodesic segment $\gamma_1 \colon [0, \lambda] \to W_1$ joining p_1 and q_1 and a unique geodesic segment $\gamma_2 \colon [0, \lambda] \to W_2$ joining p_2 and q_2. Since B_1 and B_2 are starshaped with respect to p_1 and p_2, respectively, then $\gamma_1 \subset B_1$ and $\gamma_2 \subset B_2$. Consequently, $\gamma = (\gamma_1, \gamma_2)$ is contained in $B_1 \times B_2$ which means that (p_1, p_2) sees all the points of $B_1 \times B_2$ via $B_1 \times B_2$ and so $B_1 \times B_2$ is starshaped.

Proof of Part (ii)

From the proof of the necessity and sufficiency of Part (i), we obtain that

$p_1 \in \ker B_1, p_2 \in \ker B_2 \Rightarrow (p_1, p_2) \in \ker (B_1 \times B_2)$

i.e

$$\ker B_1 \times \ker B_2 \subset \ker (B_1 \times B_2) \qquad (3.1)$$

Moreover,

$(p_1, p_2) \in \ker (B_1 \times B_2) \Rightarrow p_1 \in \ker B_1, p_2 \in \ker B_2$

i.e.

$$\ker (B_1 \times B_2) \subset \ker B_1 \times \ker B_2 \qquad (3.2)$$

From (3.1) and (3.2) we have that

$$\ker (B_1 \times B_2) = \ker B_1 \times \ker B_2.$$

The following example is an application of Proposition (3.8).

Example (3.9)

Consider the starshaped subsets $B_1 = I_1 \cup I_2 \subset E^2$ and $B_2 = [0, 1] \subset E^1$, where $I_1 = \{(x, 0) \colon 0 \leq x \leq 1\}$ and $I_2 = \{(0, y) \colon 0 \leq y \leq 1\}$. Clearly, $\ker B_1 = (0, 0)$ and $\ker B_2 = B_2$. Fig. (3) shows that $B_1 \times B_2$ is starshaped with respect to any point of the set

$K = \{(0, 0, z) \colon 0 \leq z \leq 1\} = \ker (B_1 \times B_2).$

At the same time

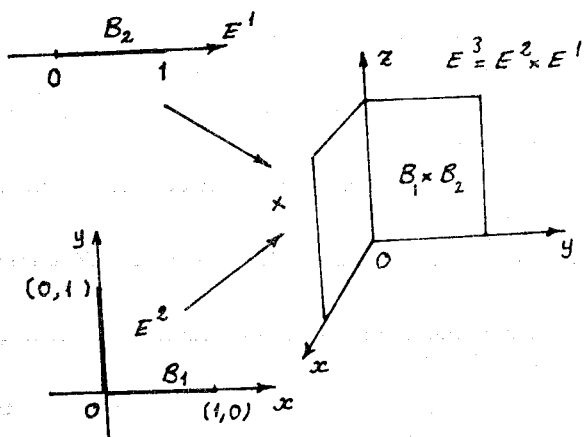$K = \{(0, 0)\} \times B_2 = \ker B_1 \times \ker B_2.$

Fig. (3)

Similar to Corollary (3.4), we can prove that a starshaped subset $B \subset W_1 \times W_2$ has starshaped natural projections but the converse is not necessarily true. Circular annulus in E^2 can be considered as an example indicating the last claim.

4. CONCLUSION

(a) All results of the present work hold in the cartesian product of Euclidean as well as hyperbolic spaces as this sort of manifolds represents nice examples of complete simply connected C^∞ Riemannian manifolds without conjugate points. Also the results we have just proved are valid in the case of cartesian product of manifolds without focal points as every manifold without focal points has no conjugate points.

(b) If another Riemannian metric and connection are given on $M_1 \times M_2$ instead of g and D mentioned in Section (2), results of this paper will be changed.

(c) The problem which is more interesting is to consider the convexity and starshape concepts in the cartesian product of general Riemannian manifolds.

(d) The study we have established in this work could be considered as a base of a study of other concepts such as local convexity, local non-convexity, supporting subsets, separating subsets, ..., etc.

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