

ON THE SECOND TYPE LINEAR VOLTERRA INTEGRAL EQUATIONS WITH CONVOLUTION KERNEL

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ABSTRACT

In this study, sufficient conditions have been given to obtain absolute monotonicity solution for the second type linear Volterra integral equations with convolution kernel having unity as source term. Moreover, positive and negative solutions have been obtained in accordance with the theorem for the equations including monotonic source term.

1. INTRODUCTION

Let source term, kernel function and unknown function be denoted by φ , K and f , respectively. Let the following equation be given,

$$f(t) = \varphi(t) + \lambda \int_0^t K(t, \tau) f(\tau) d\tau. \quad (1.1)$$

This is a linear Volterra integral equation. The following three theorems about this equation are well known.

Theorem 1.1. Let K is quadratically integrable in the square $\{(t, \tau): 0 \leq t \leq T, 0 \leq \tau \leq T\}$, where $T > 0$ is a constant and $\varphi \in L_2[0, T]$. Then, there exists a unique L_2 -solution $f(t)$ of the equation (1.1) which is given by Neumann series as

$$f(t) = \sum_{n=0}^{\infty} \lambda^n \Psi_n(t) \quad (1.2)$$

Where

$$\Psi_0(t) = \varphi(t), \quad \Psi_n(t) = \int_0^t K_n(t, \tau) \varphi(\tau) d\tau, \quad n = 1, 2, 3, \dots \quad (1.3)$$

and

$$K_1(t, \tau) = K(t, \tau),$$
$$K_{n+1}(t, \tau) = \int_{\tau}^t K(t, s) K_n(s, \tau) ds, \quad n = 1, 2, 3, \dots \quad (1.4)$$

[Tricomi (1957)].

The following result can be concluded from this theorem.

Result 1.1. If λ and K have the same sign, then the sign of the solution of $f(t)$ is the same as that of φ .

The second type linear Volterra integral equation with convolution kernel is given as follows

$$f(t) = \varphi(t) - \int_0^t K(t-\tau) f(\tau) d\tau = \varphi(t) - K*f. \quad (1.5)$$

Then the solution of (1.5) can be obtained by using the solution of the equation

$$u(t) = 1 = \int_0^t K(t-\tau) u(\tau) d\tau. \quad (1.6)$$

Theorem 1.2. (Convolution Theorem) If $\varphi'(t)$ exists for $0 \leq t \leq T$ and the conditions of $\int_0^T |\varphi'(t)| dt < \infty$ and

$\int_0^T |K(t)| dt < \infty$, are satisfied. Then the solution to (1.5) is given by

$$f(t) = u(t) \varphi(0) + \int_0^t u(t-\tau) \varphi'(\tau) d\tau \quad (1.7)$$

[Bellman (1963)].

Theorem 1.3. (Equivalence Theorems) If $K \in C^1 [0, \infty)$ and φ is locally integrable, then the following two integral equations are equivalent:

$$f(t) = \varphi(t) - \int_0^t K(t-\tau) f(\tau) d\tau \quad (1.8)$$

$$f(t) = \Psi(t) - \int_0^t L(t-\tau) f(\tau) d\tau, \quad (1.9)$$

where

$$L(t) = g'(t) + a g(t) + \int_0^t g(t-\tau) K'(\tau) d\tau, \quad (1.10)$$

and

$$\Psi'(t) = \varphi(t) + \int_0^t g'(t-\tau) \varphi(\tau) d\tau. \tag{1.11}$$

Here $a = K(0)$, g is any function such that $g \in C^1 [0, \infty)$ and $g(0) = 1$ [Ling (1978)].

Definition 1.1. A function f is said to be absolutely monotonic in an interval $[0, T]$ if $f \in C^\infty [0, T]$ and

$$\frac{d^n f(t)}{dt^n} \geq 0 \text{ for } 0 \leq t \leq T; \quad n = 0, 1, 2, \dots$$

are satisfied [Friedman (1963)].

2. STATEMENTS OF RESULTS

Theorem 2.1. The following four relations are satisfied

(1) $K(0) = a < 0$, (2) $K'(0) = b$, (3) $a^2 \geq 4b$, (4) $K^{(n)}(t) < 0$; for all $n \geq 2$ and $K \in C^n$ for $n = 0, 1, 2, \dots$. Then the solution of the equation

$$f(t) = 1 - \int_0^t K(t-\tau) f(\tau) d\tau, \quad (0 \leq t \leq T, T < \infty) \tag{2.1}$$

satisfies $f^{(n)}(t) < 0$ with $n = 0, 1, 2, \dots$. That explains f is absolutely monotonic.

Proof: The proof will be achieved for two cases.

Case 1: Let $b \leq 0$. K' is decreasing since $K''(t) < 0$ by the hypothesis (4). Due to the assumption of $K'(0) = b \leq 0$, we can easily obtain that $K'(t) < 0$. Therefore, K becomes decreasing. Since $K(0) = a < 0$ by the hypothesis (1), $K(t) < 0$. Then, the solution of the integral equation

$$f(t) = 1 - \int_0^t K(t-\tau) f(\tau) d\tau = 1 - K^* f$$

is positive by the result 1.1. (i.e. $f(t) > 0$).

Furthermore, if the equation (2.1) is n -times differentiable then,

$$f'(t) = - [K(0) f(t) + K'^* f]$$

$$f''(t) = - [K(0) f'(t) + K'(0) f(t) + K''^* f]$$

.....

$$f^{(n)}(t) = - [K(0) f^{(n-1)}(t) + K'(0) f^{(n-2)}(t) + \dots + K^{(n-1)}(0) f(t) + K^{(n)*} f], \quad (n \geq 1)$$

are obtained. By induction we can show that $f^{(n)}(t) > 0$ for $n \geq 1$ as follows:

Since $f(t) > 0$,

$$f'(t) = - [K(0) f(t) + K'^* f] > 0$$

holds. i.e. for $n = 1$ $f'(t) > 0$. Assume that $f^{(n)}(t) > 0$ for $1, 2, \dots, n$. Thus

$$f^{(n+1)}(t) = - [K(0) f^{(n)}(t) + K'(0) f^{(n-1)}(t) + K''(0) f^{(n-2)}(t) + \dots + K^{(n-1)}(0) f'(t) + K^{(n)}(0) f(t) + K^{(n+1)*} f] > 0$$

becomes valid. Thus we have completed the proof of the theorem for the case one.

Case 2: Let $K'(0) = b > 0$. According to the equivalence theorem, the equation

$$f(t) = 1 - K^* f$$

is equivalent to

$$f(t) = g(t) - L^* f,$$

where

$$L(t) = g'(t) + a g(t) + g^* K'$$

such that $g \in C^1$ and $g(0) = 1$. Now choose $g(t) = e^{-\gamma t}$ with $\gamma = \frac{1}{2} (a + \sqrt{a^2 - 4b})$.

Thus $\gamma < 0$, $a - \gamma < 0$ and $\gamma^2 - a\gamma + b = 0$ are deduced. On the other hand

$$\begin{aligned} L(t) &= (a - \gamma)e^{-\gamma t} + K'^* e^{-\gamma t} \\ \Rightarrow L'(t) &= (\gamma^2 - a\gamma + b)e^{-\gamma t} + K''^* e^{-\gamma t} \\ &= K''^* e^{-\gamma t} \end{aligned}$$

holds and the function f has the form

$$f(t) = e^{-\gamma t} - L^* f.$$

Since $L(0) = a - \gamma < 0$ and $L'(t) < 0$ so, we can easily show that $L(t) < 0$. Thus according to the result (1.1) $f(t) > 0$ becomes obvious.

Furthermore of L 's differentiated successively, the followings are obtained.

$$L'' = K''(t) - \gamma e^{-\gamma t} K''$$

$$L^{(n)} = K^{(n)}(t) - \gamma K^{(n-1)}(t) + \gamma^2 K^{(n-2)}(t) - + \dots$$

$$+ (-\gamma)^{n-2} K''(t) + (-\gamma)^{n-1} e^{-\gamma t} K'', \quad (n \geq 2).$$

If $-\gamma < 0$ and $K^{(n)}(t) < 0$ are taken into consideration then $L^{(n)}(t) < 0$ for all $n \geq 2$. If we recall that $L(t) < 0$ and $L'(t) < 0$, then $L^{(n)}(t) < 0$, $n = 0, 1, 2, \dots$

Let us obtain the successive derivatives of the function

$$f(t) = e^{-\gamma t} - L * f = e^{-\gamma t} - f * L.$$

These are in the following

$$f'(t) = -\gamma e^{-\gamma t} - f(0) L(t) - f' * L,$$

$$f^{(n)}(t) = (-\gamma)^n e^{-\gamma t} - f(0) L^{(n-1)}(t) - f'(0) L^{(n-2)}(t) - \dots$$

$$- f^{(n-2)}(0) L'(t) - f^{(n-1)}(0) L(t) - f^{(n)*} L, \quad (n \geq 1).$$

In this case, each of them is an integral equation for all $n \geq 1$. We can also see that $f^{(n)}(t) > 0$. The statement of

$$f'(t) = -\gamma e^{-\gamma t} - f(0) L(t) - f' * L > 0$$

holds. Now assume that $f^{(n)}(t) > 0$ for $1, 2, \dots, n$. So

$$f^{(n+1)}(t) = (-\gamma)^{n+1} e^{-\gamma t} - f(0) L^{(n)}(t) - f'(0) L^{(n-1)}(t) - f''(0) L^{(n-2)}(t) -$$

$$- f^{(n-1)}(0) L'(t) - f^{(n)}(0) L(t) - f^{(n+1)*} L$$

equation is an integral equation with $f^{(n+1)}(t)$ is unknown. If $-\gamma < 0$, $f^{(n)}(t) < 0$ for $0, 1, 2, \dots, n$ and $L^{(n)}(t) < 0$ for $0, 1, 2, \dots, n$ are remembered then source term is positive and therefore $f^{(n+1)}(t) > 0$. Previously we had proved that $f(t) > 0$. Thus $f^{(n)}(t) > 0$ for $n > 0$, $n = 0, 1, 2, \dots$

This completes the proof of the theorem.

Example 2.1. If $K(t) = -e^t$ is chosen in equation (2.1) then the solution becomes

$$f(t) = \frac{1}{2} + \frac{1}{2} e^{2t} \text{ with } f^{(n)}(t) > 0 \text{ for } n = 0, 1, 2, \dots,$$

A special case of the theorem for $n = 0, 1, 2, 3$ had been proved by Rina Ling [Ling (1978)].

Theorem 2.2 In addition to the assumptions of Theorem (2.1), Let $\varnothing \in C^1$ in the form of the equation

$$f(t) = \varnothing(t) - \int_0^t K(t-\tau) f(\tau) d\tau \quad (2.2)$$

Then

- (1) If $\varnothing(0) < 0$ and $\varnothing'(t) > 0$ then $f(t) > 0$ (i.e. f has no root).
- (2) If $\varnothing(0) < 0$ and $\varnothing'(t) < 0$ then $f(t) < 0$ (i.e. f has no root).

Proof: Let

$$h(t) = 1 - \int_0^t K(t-\tau) h(\tau) d\tau.$$

By the convolution theorem, the solution of $f(t)$ of the equation (2.2) becomes

$$f(t) = h(t) \varnothing(0) + \int_0^t h(t-\tau) \varnothing'(\tau) d\tau.$$

According to the Theorem (2.1) we can obtain that $h^{(n)}(t) > 0$ holds for $n = 0, 1, 2, \dots$

Suppose that the conditions of (1) are satisfied.

$f(0) = h(0) \varnothing(0) > 0$ and also

$$f'(t) = h'(t) \varnothing(0) + h(0) \varnothing'(t) + \int_0^t h'(t-\tau) \varnothing'(\tau) d\tau < 0$$

is satisfied, then $f(t) > 0$.

Suppose that the conditions of (2) are satisfied. Then, $f(t) < 0$ holds since $f(0) < 0$ and $f'(t) < 0$.

ÖZET

Bu çalışmada birim kaynaklı ikinci tip lineer konvolüsyon çekirdekli Volterra integral denkleminin mutlak monoton bir çözüme sahip

olmasının yeter şartları verildi. Ayrıca bu teoreme bağlı olarak monoton bir kaynağa sahip denklem için pozitif ve negatif çözümler elde edildi.

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