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ON THE DEGREE OF APPROXIMATION OF A FUNCTION BY NÖRLUND MEANS OF ITS FOURIER - JACOBI SERIES

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ABSTRACT

In the present paper we prove a theorem on the degree of approximation of a function by Nörlund means of its Fourier-Jacobi series, which generalizes the results of [2] and [3].

1. Let $\Sigma\ a_n$ be any given series with the sequence of partial sums $\{S_n\}_{n=1}$ If $\{P_n\}$ is a sequence of constants, real or complex numbers, such that

$$P_n = p_0 + p_1 \cdots\cdots + p_n \tag{1.1}$$

then the sequence–to–sequence–transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v\ S_{n-v} \tag{1.2}$$

defines the sequence $\{t_n\}$ of Nörlund means of the series $\sum\limits_{n=0}^{\infty} a_n$ generated by the sequence $\{p_n\}$.

The series $\sum\limits_{n=0}^{\infty} a_n$ is said be summable by Nörlund means or summable (N, p_n) to the sum S, if limit t_n exists and equal to S as$n \to \infty$.

2. Let $F(\theta) = f(\text{Cos}\ \theta)$, $\theta \in [0, \pi]$be a Lebesque measurable function such that

$$\int_0^{\pi} f(\theta)\ p_n\,(\text{Cos}\ \theta)\ (\text{Sin}\ \theta)^{2\beta+1}\ \text{Cos}\ \theta)^{2\beta+1}\ d\theta$$

exists, where $\beta > -1$, and p_n (Cos θ) is the nth–Jacobi polynomial of order (α, β). The Fourier– Jacobi series associated with this function is given by

$$f(\Theta) \sim \sum_{n=1}^{\infty} \hat{f}(n)\ h_n\ R_n(\text{Cos}\ \theta) \tag{2.2}$$

where

$$\hat{f}(n) = \int_0^{\pi} f(\varnothing)\ R_n(\text{Cos}\ \theta)\ d_{\mu}(\varnothing) \tag{2.3}$$

$$h_n = \int_0^{n} [R_n(\text{Cos}\ \theta)\ d_{\mu}(\theta)\ ]^{-1}$$

$$= \frac{\gamma(2n + \alpha + \beta\ 1)\ \gamma(n + \beta + \alpha + 1)\ \gamma(n + \beta + 1)}{(n + \beta + 1)\ \gamma(n + \beta)\ \gamma(\alpha + 1)\ \gamma(\beta + 1)} \tag{2.4}$$

$$R_n(\text{Cos}\ \theta) = \frac{p_n(\text{Cos}\ \theta)}{P_n(1)} \tag{2.5}$$

and

$$d\mu(\theta) = (\text{Sin}\ (\theta)^{2\alpha + 1}\ (\text{Cos}\ \theta)^{2\beta + 1} \tag{2.6}$$

Askey and Wainger [1] have defined the convolution structure of two functions f_1 and f_2 of L–class on $[0, \pi]$ in the following manner:

$$(f_1{}^{*}\ f_2)\ (\theta) = \int_0^{\pi} f_1(\varnothing)\ T_{\varnothing}\ f_2(\theta)\ d_{\mu}(\varnothing) \tag{2.7}$$

where the generalisation translation $T_{\varnothing}$ is defined by

$$T_{\varnothing}(\theta) = \int_0^{\pi} f(\psi)\ k(\theta, \varnothing, \psi)\ d_{\mu}(\psi) \tag{2.8}$$

and $K(\theta, \varnothing, \psi)$ is a non–negative symmetric function such that

$$R_n(\text{Cos}\ \theta)\ R_n(\text{Cos}\ \varnothing) = \int_0^{\pi} k(\theta, \varnothing, \psi)\ (\text{Cos}\ \psi)\ d_{\mu}(\psi) \tag{2.9}$$

$$\int_0^{\pi} k(\theta, \varnothing, \psi)\ d_{\mu}(\psi) = 1 \tag{2.10}$$

3. Partial sum $S_n(f;\theta)$ of the series (2.2) is given by

$$S_n(f;\theta) = \sum_{v=0}^{n} \hat{f}(v)\, h_v\, R_v(\mathrm{Cos}\,\theta)$$

$$= \sum_{v=0}^{n} h_v \int_0^{\pi} f(\varnothing)\, R_v(\mathrm{Cos}\,\theta)\, R_v(\mathrm{Cos}\,\theta)\, d_{\mu}\varnothing$$

Now using the orthogonal property of Jacobi polynomials and the relation (2.9) we have

$$S_n(f;\theta) - f(\theta) = \sum_{v=0}^{n} h_v \int_0^{\pi} f(\varnothing)\, k(\theta,\varnothing,\psi) R_n(\cos\psi)\, d_{\mu}(\varnothing)\, d_{\mu}(\psi) f(\theta)$$

$$= \sum_{v=0}^{n} h_v \int_0^{\pi} \{T_{\varnothing} f(\theta) - f(\theta)\}\; R_n\,\mathrm{Cos}\,\psi)\, d_{\mu}(\psi)$$

$$= B_n \int_0^{\pi} w_f(\psi)\, R_n(\mathrm{Cos}\,\psi)\, d_{\mu}(\psi) \tag{3.1}$$

where $\omega_f(\psi) = T_{\varnothing}(f(\theta) - f(\theta))$ (3.2)

and $$B_n = \frac{\gamma(n+\beta+\alpha+2)}{\gamma(\alpha+1)\,\gamma(n+\beta+1)} \sim n^{\alpha+1}.$$

Therefore, we have

$$t_n(\theta) - f(\theta) = \frac{1}{P_n} \sum_{k=0}^{n} p_k\, \{S_{n-k}(f;\theta) - f(\theta)\}$$

$$= \frac{1}{P_k} \sum_{k=0}^{n} p_k\, B_{n-k} \int_0^{\pi} \omega(\psi)\, p_{n-k}^{(\alpha+1,\,\beta)}(\mathrm{Cos}\,\psi)\, d\psi$$

where $$\omega(\psi) = \omega_f(\psi) \left(\mathrm{Sin}\,\frac{\psi}{2}\right)^{2\alpha+1} \left(\mathrm{Cos}\,\frac{\psi}{2}\right)^{2\beta+1}$$

In 1986 Pandey [3] proved the following theorem

Theorem A. Let $0 < \delta \leq \lambda$, If x is a point such that

$$\varnothing(t) = \int_t^{\delta} \frac{|\varnothing(u)|}{u}\; {}^{p}\!\left[\frac{1}{n}\right] du$$

$$= O\left(P\left[\frac{1}{t}\right] g(t) \right) \text{ as } t \to 0$$

then

$$t_n(x) - f(x) = O\left(g\left(\frac{1}{n}\right)\right)$$

where g (t) is a positive increasing function such that

$$P\left[\frac{1}{t}\right] g(t) \to \infty \text{ as } t \to 0$$

In (1988) Pathak and Jain [2] proved the following theorem

Theorem B: If $\{p_n\}$ is a non–negative and non–increasing sequence of real or complex numbers, $-1 \leq \alpha \leq -\frac{1}{2}$, $\beta > \alpha$ and

$$\int_t^\delta \frac{\omega(\mu)\, p_c\left(\frac{1}{u}\right)}{u^{\alpha+3/2}}\, du = O(1) \text{ as } t \to 0$$

$$\text{then } L_n(f;\theta) - f(\theta) = O\left(\frac{1}{P_n}\right).$$

The object of the present paper is to generalize the above two theorems A and B in following from.

Our theorem is as follows:

Theorem: If $\{p_n\}$ is a non–negative and non–increasing sequence of real or complex numbers, $-1 \leq \alpha \leq -\frac{1}{2}$, $\beta > \alpha$ and

$$\Phi(t) = \int_t^\delta \frac{w(u)\, P\left(\frac{1}{u}\right)}{u^{\alpha} + \frac{3}{2}}\, du = O\left(P_{\left[\frac{1}{t}\right]} g(t)\right). \tag{4.1}$$

as $t \to 0$

then

$$L_n(f;\theta) - f(\theta) = O\left(g\left(\frac{1}{n}\right)\right)$$

where g (t) is a positive, increasing function such that

$$P_{\left[\frac{1}{t}\right]}\, g(t) \rightarrow \infty \text{ as } t \rightarrow 0. \tag{4.2}$$

We shall use the following lemmas in the proof of our theorem.

Lemma 1: [2]: Let α, β be real numbers or equal to -1, then

$$\text{for } 0 \leq \psi \leq \frac{1}{n}$$

$$N_n(\psi) = O(n^{2\alpha+2})$$

where

$$N_n(\psi) = \frac{1}{P_n} \sum_{k=0}^{n} p_k\, B_{n-k}\, P_{n-k}^{(\alpha+1,\beta)}(\cos\psi)$$

Lemma 2: [5]: For $\frac{1}{n} \leq \psi \leq \pi - \frac{1}{n}$,

$$N_n(\psi) = O\left(\frac{n^{\alpha-\frac{1}{2}}}{p_n}\right)\left(P\left(\frac{1}{\psi}\right)(\operatorname{Sin}\psi/2)^{\alpha+\frac{3}{2}}\,(\cos\psi/2)^{\beta+\frac{1}{2}}\right.$$

$$+\; O\left[\left(n^{\alpha-\frac{1}{2}}\right)\left\{(\operatorname{Sin}\psi/2)^{-\alpha-5/2}\,(\operatorname{Cos}\psi/2)^{-\beta-\frac{3}{2}}\right\}\right]$$

Lemma 3: Under the condition (4.1), we have

$$\int_0^t |\omega(u)|\, du = O\left(t^{\alpha+\frac{3}{2}}\, g(t)\right) \tag{5.3}$$

Proof of Lemma 3: Let

$$\omega(t) = \int_0^t |\omega(u)|\, P_{\left[\frac{1}{u}\right]}\, du$$

using the condition (4.1), we have

$$\int_0^t u\, \Phi'(u)\, du = \int_0^t |\, \omega(u)\, |\, P_{\left[\frac{1}{u}\right]}\, du$$

on integrating by parts, we get

$$\omega(t) = O\left(t^{\alpha+\frac{3}{2}} P_{\left[\frac{1}{t}\right]} g(t)\right) + \int_0^t P_{\left[\frac{1}{u}\right]} g(u)\, du$$

$$= O\left(t^{\alpha+\frac{3}{2}} P_{\left[\frac{1}{t}\right]} g(t)\right)$$

Thus we have $\displaystyle\int_0^t |\, \omega(u)\, |\, du = \int_0^t \frac{|\, w(u)\, |\, P_{\left[\frac{1}{u}\right]}\, du}{P_{\left[\frac{1}{u}\right]}}$

$$\leq \frac{1}{P_{\left[\frac{1}{t}\right]}} \int_0^t |\, \omega(u)\, |\, P_{\left[\frac{1}{u}\right]}\, du$$

$$= O\left(\frac{1}{P_{\left[\frac{1}{t}\right]}}\right) O\left(t^{\alpha+\frac{3}{2}} \mathrm{xp}_{\left[\frac{1}{t}\right]} g(t)\right) = O\left(t^{\alpha+\frac{3}{2}} g(t)\right)$$

Proof of the Theorem: We have

$$I_n(f; \theta) - f(\theta) = \int_0^{\pi} \omega(\psi)\, N_n(\psi)\, d(\psi)$$

$$= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi} \omega(\psi)\, N_n(\psi)\, d\psi$$

$$= I_1 + I_2 + I_3, \text{ say}$$

we have

$$I_1 = \int_0^{\frac{1}{n}} \omega(\psi)\, N_n(\psi)\, d\psi$$

$$= O\left(n^{2\alpha+2}\right) O\left(\frac{1}{n^{2+\frac{3}{2}}}\, g\left(\frac{1}{n}\right)\right)$$

$$= O\left(g\left(\frac{1}{n}\right)\right) \quad \alpha < -\frac{1}{2}$$

Now, we consider I_3

$$I_3 = \int_{\pi-\frac{1}{n}}^{\pi} \omega(\psi)\, N_n(\psi)\, d\psi$$

$$= \int_0^{\frac{1}{n}} \omega(\pi-\psi) \qquad N_n(\psi)\, d\psi$$

$$= O\left(n^{2\alpha+2}\right) \int_0^{\frac{1}{n}} |\, \omega(\pi-\psi)\, |\, d\psi$$

$$= O\left(n^{2\alpha+2}\right) O\left(\frac{1}{n^{\alpha+\frac{3}{2}}}\, g\left(\frac{1}{n}\right)\right)$$

$$= O\left(g\left(\frac{1}{n}\right)\right)$$

Lastly, we consider I_2

$$I_2 = \int_{\frac{1}{n}}^{\pi-\frac{1}{n}} \omega(\psi)\, N_n(\psi)\, d\psi$$

$$= O\left(\frac{n^{\alpha-\frac{1}{2}}}{P_n}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} |\omega(\psi)|\, p\left(\frac{1}{\psi}\right)(\operatorname{Sin}\psi/2)^{-\alpha-\frac{3}{2}}(\omega\,\psi/2)^{-\beta-\frac{1}{2}}\, d\psi$$

$$+\ O\left(n^{\alpha-\frac{1}{2}}\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} |\omega(\psi)|\left\{(\operatorname{Sin}\psi/2)^{-\alpha-\frac{5}{2}}\left(\operatorname{Cos}\frac{\psi}{2}\right)^{-\beta-\frac{3}{2}}\right\}d\,\psi\right.$$

$$= O\left(\frac{n^{\alpha+\frac{1}{2}}}{P_n}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} \frac{|\omega(\psi)|\ P\left(\frac{1}{\psi}\right)d\,\psi}{\psi^{\alpha+\frac{3}{2}}}$$

$$+\ O\left(\frac{n^{\alpha-\frac{1}{2}}}{P_n}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} \frac{|\omega(\psi)|\ p\left(\frac{1}{\psi}\right)d\,\psi}{\psi^{\alpha+\frac{5}{2}}}$$

$$= O\left(\frac{1}{P_n}\, p_n\, g\left(\frac{1}{n}\right)\right) = O\left(g\left(\frac{1}{n}\right)\right)$$

combining the relations I_1, I_2, I_3, we get

$$I_n\ (f;\ \theta) - f\ (\theta) = O\left(g\left(\frac{1}{n}\right)\right)$$

This completes the proof of the theorem.

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