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SOME GENERATING FUNCTIONS OF MODIFIED JACOBI POLYNOMIALS*

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ABSTRACT

The present note deals with the derivation of novel extensions of some bilateral and mixed trilateral generating functions of modified Jacobi polynomials $P_n^{(\alpha, \beta-n)}(x)$ from the stand point of one parameter group of continuous transformations. Some particular cases of interest as wel as the applications of our results are laid down.

1. INTRODUCTION

In recent years various properties of Jacobi polynomials defined in [11],

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_2F_1 \left[\begin{matrix} -n, \ 1+\alpha+\beta+n \ ; & \frac{1-x}{2} \\ 1+\alpha \ ; & \end{matrix} \right] \quad (1.1)$$

have been found extensively studied in [1, 2, 8, 9, 10] using group-theoretic method introduced by L. Weisner [12]. In [7], it has been pointed out why Weisner's method (with the suitable interpretation of n the index) fails in the study of $P_n^{(\alpha, \beta)}(x)$. In [3], the present author studied the many properties of $P_n^{(\alpha, \beta-n)}(x)$ – the modified form of Jacobi polynomials by using Weisner's method (with the suitable interpretation of n). For various works on $P_n^{(\alpha, \beta-n)}(x)$ see [4, 5].

The aim at presenting this note is to state and establish some results on the extensions of bilateral and mixed trilateral generating

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functions of $P_n^{(\alpha, \beta-n)}(x)$ from the standpoint of one parameter group of continuous transformations. The main results of the note are stated in the form of the following theorems.

Theorem 1. If there exists a generating relation of the form:

$$G(x, w) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha, \beta-n)}(x) w^n \tag{1.2}$$

then

$$(1-w)^{\beta} \left\{1-\frac{w}{2}(1+x)\right\}^{-1-\alpha-\beta-m} G\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)}, \frac{wz}{1-w}\right)$$

$$= \sum_{n=0}^{\infty} w^n g_n(z) P_{n+m}^{(\alpha, \beta-n)}(x) \tag{1.3}$$

where

$$g_n(z) = \sum_{p=0}^{n} a_p \frac{(m+p+1)_{n-p}}{(n-p)!} z^p.$$

Theorem 2. If there exists a generating relation of the form:

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha, \beta-n)}(x) q_n(u) w^n, \tag{1.4}$$

where $q_n(u)$ is an arbitrary polynomial of degree n, then

$$(1-w)^{\beta} \left\{1-\frac{w}{2}(1+x)\right\}^{-1-\alpha-\beta-m} G\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)}, u, \frac{wz}{1-w}\right)$$

$$= \sum_{n=0}^{\infty} w^n g_n(u, z) P_{n+m}^{(\alpha, \beta-n)}(x) \tag{1.5}$$

where

$$g_n(u, z) = \sum_{p=0}^{n} a_p \frac{(m+p+1)_{n-p}}{(n-p)!} q_p(u) z^p.$$

The importance of the above theorems lies in the fact that whenever there exists generating relation of the form (1.2) (or (1.4)), the corresponding bilateral (or trilateral) generating relation can at once be written down from (1.3) (or (1.5)). Thus, one can get a large number of bilateral (or trilateral) generating relations by attributing different values to a_n in Theorem 1 (or Theorem 2).

Proof of Theorem 1. At first we define the following partial differential operator:

$$R = (1-x^2)\, y \frac{\partial}{\partial x} - 2y^2 \frac{\partial}{\partial y} - [(1+\alpha+\beta+m)(1+x)-2\beta]\, y$$

such that

$$R\left(P_{n+m}^{(\alpha,\,\beta-n)}(x)\, y^n\right) = -2\,(n+m+1)\, P_{n+m+1}^{(\alpha,\,\beta-n-1)}(x)\, y^{n+1}. \tag{1.6}$$

The extended form of the group generated by R is

$$e^{wR} f(x, y) = (1+2wy)^{\beta}\, \{1+wy(1+x)\}^{-1-\alpha-\beta-m} \times f\left(\frac{x+wy(1+x)}{1+wy(1+x)},\ \frac{y}{1+2wy}\right). \tag{1.7}$$

In the formula

$$G(x, w) = \sum_{n=0}^{\infty} a_n\, P_{n+m}^{(\alpha,\,\beta-n)}(x)\, w^n,$$

replacing w by wyz and then operating both sides by exp (wR)(we get

$$e^{wR}\, G(x, wyz) = e^{wR} \sum_{n=0}^{\infty} a_n\, (wz)^n \left(P_{n+m}^{(\alpha,\,\beta-n)}(x)\, y^n\right). \tag{1.8}$$

The left hand member of (1.8) is

$$(1+2wy)^{\beta}\, \{1+wy(1+x)\}^{-1-\alpha-\beta-m}\, G\left(\frac{x+wy(1+x)}{1+wy(1+x)},\ \frac{wyz}{1+2wy}\right). \tag{1.9}$$

The right hand member of (1.8) is

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n\, (wz)^n\, \frac{w^k}{k!}\, R^k \left(P_{n+m}^{(\alpha,\,\beta-n)}(x)\, y^n\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n (wz)^n \frac{w^k}{k!} (-2)^k (n+m+1)_k P_{n+m+k}^{(\alpha, \beta-n-k)} (x) y^{n+k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-2wy)^n \left(a_{n-k} (-z/2)^{n-k} \frac{(n-k+m+1)_k}{k!} \right) P_{n+m}^{(\alpha, \beta-n)} (x) \qquad (1.10)$$

Equating (1.9) and (1.10) and then replacing 2wy by "$-w$" and $(-z/2)$ by "z" we get

$$(1-w)^{\beta} \left\{1- \frac{w}{2} (1+x)\right\}^{-1-\alpha-\beta-m} G\left(\frac{x- \frac{w}{2} (1+x)}{1- \frac{w}{2} (1+x)}, \frac{wz}{1-w} \right)$$

$$= \sum_{n=0}^{\infty} w^n g_n(x) P_{n+m}^{(\alpha, \beta-n)} (x),$$

where

$$g_n(z) = \sum_{p=0}^{n} a_p \frac{(p+m+1)_{n-p}}{(n-p)!} z^p$$

this completes the proof of the Theorem 1.

Putting $m = 0$ in Theorem 1, we get

Corollary 1. If

$$G(x, w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha, \beta-n)} (x) w^n$$

then

$$(1-w)^{\beta} \left\{ 1- \frac{w}{2} (1+x)\right\}^{-1-\alpha-\beta} G\left(\frac{x- \frac{w}{2} (1+x)}{1- \frac{w}{2} (1+x)}, \frac{wz}{1-w} \right)$$

$$= \sum_{n=0}^{\infty} w^n g_n (z) P_n^{(\alpha, \beta-n)} (x)$$

where

$$g_n(z) = \sum_{p=0}^{n} a_p \frac{(p+1)_{n-p}}{(n-p)!} z^p,$$

which is found derived in [4].

Proof of the Theorem 2. Let us first consider the formula,

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha, \beta-n)}(x)\, q_n(u)\, w^n.$$

Replacing w by wyz and then operating both sides by (exp w R) we get

$$(\exp wR)\, G(x, u, wyz) = (\exp wR) \sum_{n=0}^{\infty} a_n (wz)^n q_n(u) (P_{n+m}^{(\alpha, \beta-n)}(x) y^n)$$

which on simplification reduces to

$$(1-w)^{\beta} \left\{1 - \frac{w}{2}(1+x)\right\}^{-1-\alpha-\beta-m} G\left(\frac{x - \frac{w}{2}(1+x)}{1 - \frac{w}{2}(1+x)}, u, \frac{wz}{1-w}\right)$$

$$= \sum_{n=0}^{\infty} w^n g_n(u, z) P_{n+m}^{(\alpha, \beta-n)}(x)$$

where

$$g_n(z, u) = \sum_{p=0}^{n} a_p \frac{(n+m+1)_{n-p}}{(n-p)!} q_p(u) z^p,$$

which completes the proof of the Theorem 2.

Putting m = 0 in Theorem 2, we get

Corollary 2. If

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha, \beta-n)}(x)\, q_n(u)\, w^n$$

then

$$\sum_{n=0}^{\infty} w^n g_n(u, z) P_n^{(\alpha, \beta-n)}(x) = (1-w)^{\beta} \left\{1 - \frac{w}{2}(1+x)\right\}^{-1-\alpha-\beta}$$

$$\times\ G\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)},\ u,\ \frac{wz}{1-w}\right)$$

where

$$g_n(u, z) = \sum_{p=0}^{n} a_p\ q_p(u)\ \frac{(n+1)_{n-p}}{(n-p)!}\ z^p.$$

Before discussing the applications of our results, we like to point out that due to the existence of the relation,

$$P_n^{(\alpha,\ \beta)}(x) = (-1)^n\ P_n^{(\beta,\ \alpha)}(-x)$$

we may get the following results analogous to Theorem 1 and Theorem 2.

Result 1. If

$$G(x\ w) = \sum_{n=0}^{\infty} a_n\ P_{n+m}^{(\alpha-n,\ \beta)}(x)\ w^n$$

then

$$\sum_{n=0}^{\infty} w^n\ g_n(z)\ P_{n+m}^{(\alpha-n,\ \beta)}(x) = (1+w)^{\alpha}\left\{1+\frac{w}{2}(1-x)\right\}^{-1-\alpha-\beta-m}$$

$$\times\ G\left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)},\ \frac{wz}{1+w}\right)$$

where

$$g_n(z) = \sum_{p=0}^{n} a_p\ \frac{(m+p+1)_{n-p}}{(n-p)!}\ z^p.$$

Note 1. Putting $m = 0$ in Result 1, we get the correct version of the result found derived by S. Das [6].

Result 2. If

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n P_{n+m}^{(\alpha-n, \beta)}(x) q_n(u) w^n$$

then

$$(1+w)^{\alpha}\left\{1+\frac{w}{2}(1-x)\right\}^{-1-\alpha-\beta-m} G\left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, u, \frac{wz}{1+w}\right)$$

$$= \sum_{n=0}^{\infty} P_{n+m}^{(\alpha-n, \beta)}(x) g_n(u, z) w^n$$

where

$$g_n(u, z) = \sum_{p=0}^{n} a_p \frac{(p+m+1)_{n-p}}{(n-p)!} q_p(u) z^p.$$

Note 2. If we put $m=0$, we get the correct version of the result found derived by S. Das [6].

2. APPLICATIONS

(i) Application of Theorem 1.

At first we consider the following generating relation [1]:

$$(1-w)^{\beta}\left\{1-\frac{w}{2}(1+x)\right\}^{-1-\alpha-\beta-m} P_m^{(\alpha, \beta)}\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)}\right) \quad (2.1)$$

$$= \sum_{n=0}^{\infty} \frac{(m+1)_n}{n!} P_{n+m}^{(\alpha, \beta-n)}(x) w^n.$$

If we take $a_n = \frac{(m+1)_n}{n!}$, then

$$G(x, w) = (1-w)^{\beta}\left\{1-\frac{w}{2}(1+x)\right\}^{-1-\alpha-\beta-m} P_m^{(\alpha, \beta)}\left(\frac{x-\frac{w}{2}(1+x)}{1-\frac{w}{2}(1+x)}\right).$$

Therefore applying our Theorem 1, we get

$$\{1-w(1+z)\}^{\beta}\ \{1-\frac{w}{2}(1+x)(1+z)\}^{-1-\alpha-\beta-m} \times$$

$$P_m^{(\alpha,\beta)}\left(\frac{x-\frac{w}{2}(1+x)(1+z)}{1-\frac{w}{2}(1+x)(1+z)}\right) = \sum_{n=0}^{\infty} P_{n+m}^{(\alpha,\beta-n)}(x)\ g_n(z)\ w^n \tag{2.2}$$

where

$$g_n(z) = \sum_{p=0}^{n} \binom{m+p}{p} \binom{m+n}{m+p} z^p.$$

Note 3. The relation (2.2) at $m=0$ is obtained by applying Corollary 1 on the relation (2.1) at $m = 0$.

(ii) Application of Corollary 2.

We now consider the following generating relation [10]

$$(1-w)^{b+\alpha+\beta+1}\ \{1-(1+u)\frac{w}{2}\}^{-a-b-1}\ \{1-(1+x)\frac{w}{2}\}^{-\alpha-\beta-1}$$

$$\times F_1\left(1+\alpha+\beta,\ \alpha-a,\ 1+a+b,\ 1+\alpha;\ \frac{(1-x)\frac{w}{2}}{1-(1+x)\frac{w}{2}},\right. \tag{2.3}$$

$$\left.\frac{(1-x)(1-u)\frac{w}{4}}{(1-(1+x)\frac{w}{2})(1-(1+u)\frac{w}{2})}\right) = \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta-n)}(x)\ P_n^{(\alpha,\beta-n)}(u) w^n$$

If we take $a_n = \frac{n!}{(1+\alpha)_n}$, $q_n(u) = P_n^{(\alpha,\beta-n)}(u)$, then

$$G(x,u,w) = (1-w)^{b+\alpha+\beta+1}\{1-(1+u)\frac{w}{2}\}^{-a-b-1}\{1-(1+x)\frac{w}{2}\}^{-\alpha-\beta-1}$$

$$\times F_1\left(1+\alpha+\beta,\ \alpha-a,\ 1+a+b,\ 1+\alpha;\ \frac{(1-x)\frac{w}{2}}{1-(1+x)\frac{w}{2}},\right. \tag{2.4}$$

$$\left.\frac{(1-x)(1-u)\frac{w}{4}}{\left(1-(1+x)\frac{w}{2}\right)\left(1-(1+u)\frac{w}{2}\right)}\right).$$

Therefore applying the Corollary 2, we get

$$(1-w)^{a-\alpha}\ \{1-w(1+z)\}^{b+\alpha+\beta+1}\ \left[1-w\left\{1+(1+u)\frac{z}{2}\right\}\right]^{-a-b-1}$$

$$\times\left\{1-\frac{w}{2}(1+x)(1+z)\right\}^{-\alpha-\beta-1}\ F_1\left(1+\alpha+\beta,\ \alpha-a,\ 1+a+b,\ 1+\alpha;\right.$$

$$\left.\frac{(1-x)\,wz}{2(1-w)\{1-\frac{w}{2}(1+x)(1+z)\}},\ \frac{(1-x)(1-u)\,wz}{4(1-\frac{w}{2}(1+x)(1+z))(1-w\{1+(1+u)\frac{z}{2}\})}\right)$$

$$=\sum_{n=0}^{\infty} g_n(u,z)\, P_n^{(\alpha,\ \beta-n)}(x)\, w^n \tag{2.5}$$

where

$$g_n(u,z)=\sum_{p=0}^{n}\frac{p!}{(1+\alpha)_p}\binom{n}{p}P_p^{(a,\ b-p)}(u)\, z^p.$$

Similarly our Corollary 2 may be applied to generalise the result [5]

$$\sum_{n=0}^{\infty}\frac{n!\,w^n}{(1+\alpha)_n}P_n^{(\alpha,\ \beta-n)}(x)\,L_n^{(\nu)}(u)$$

$$=\exp\left(-uw/(1-w)\right)(1-w)^{\alpha+\beta-\nu}\left(1-\frac{1}{2}(1+x)\,w\right)^{-1-\alpha-\beta}$$

$$\times \Phi_1 \left(1+\alpha+\beta;\ a-\nu,\ a+1;\ \frac{w\,(1-x)}{2-w-xw},\ \frac{(1-x)\ uw}{(2-w-xw)\ (1-w)}\right) \tag{2.6}$$

where

$$\Phi_1\,(\alpha;\,\beta,\,\gamma;\,x,\,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\beta)_n}{(\gamma)_{m+n}\ m!\ n!}\ x^n\,y^m.$$

REFERENCES

1. A.B. CHAKRABORTI., Group-theoretic origins of certain generating functions of Jacobi polynomials, J. Indian Inst. Sci. 64(B) (1983), 79–103.

2. A.K. CHONGDAR., Group-theoretic study for certain generating functions, Bull. Cal. Math. Soc. 77 (1985), 151–157.

3. A.K. CHONGDAR., On Jacobi polynomials, Bull. Cal. Math. Soc., 78 (1986), 363–374,

4. A.K. CHONGDAR and S.K. CHATTERJEA., On a class of trilateral generating relation. with Tchebicheff polynomials from the view point of one parameter group of continuous transformations, Bull. Cal. Math. Soc., 73 (1981), 127–140.

5. M.K. DAS., Some properties of special functions derived from the theory of continuous transformation groups, Proc. Amer. Math. Soc., 35 (2) (1972), 365–373.

6. S. DAS., On partial differential operators for Jacobi polynomials, Pure Math. Manuscript 3 (1984), 95–104.

7. B. GHOSH., Raising and lowering operators for classical orthogonal polynomials, Bull. Cal. Math. Soc. 76 (1984), 276–279.

8. B. GHOSH., Group-theoretic study of certain generating functions of Jacobi polynomials I, Pure Math. Manuscript, 3 (1984), 139–14;.

9. B. GHOSH., Group-theoretic study of certain generating functions of Jacobi polynomials–II to appear in Pure Math. Manısscript 5 (1986).

10. B.K. GUHATHAKURTA., Some generating functions of Jacobi polynomials, Proc. Indian Acad. Sci. (Math. Sci.) 95(1) (1986), 53–59.

11. E.D. RAINVILLE., Special Functions, Chelsea Publishing Co. Bronx., New York (1960).

12. L. WEISNER., Group-theoretic origin of certain generating function, Pacific Jour. Math. 5 (1955), 1033–1039.