Commun. Fac. Sci. Univ. Ank. Series A,
V. 41. pp. 145-150 (1992)

NOTE ON THE EXISTENCE OF F - PERFECT MORSE FUNCTIONS ON COMPACT SURFACES

DORIN ANDRICA.,

(Received April 17, 1992; Accepted July 14, 1992)

"Babeş–Bolyai" University, Faculty of Mathematics, R–3400 *Cluj–Napoca, Romania.*

ABSTRACT

This paper is dealing with the existence of F–perfect Morse functions on a smooth, compact, connected manifold, without boundary. Some concrete results concering the surfaces Tg and Pg, of genus $g \geqq 0$, are given.

PRELIMINARIES

Let M^m be a smooth compact, connected manifold of dimension $m \geq 1$, without boundary (i.e. $\partial M = \varnothing$), and let $\mathcal{F}_m(M)$ be the set of all Morse functions defined on M. For $f \in \mathcal{F}_m(M)$ let us denote by $\mu_k(f)$ the number of the critical points of f with the Morse index k, $0 \leq k \leq m$. Let $\mu(f)$ be the total number of critical points of f, i.e.

$$\mu(f) = \sum_{k=0}^{m} \mu_k(f). \tag{1}$$

The number defined by

$$\gamma(M) = \min \{\mu(f)\colon f \in \mathcal{F}_m(M)\} \tag{2}$$

is called the **Morse-Smale characteristic** of M. For more details concerning the above notins we refer to the author's book

[1, Chapter 4].

Because M^m is a compact manifold it follows that M has the homotopy type of a finite CW– complex. Therefore the singular homotopy groups $H_k(M; Z)$, $k = \overline{0, m}$, are finitely generated (see **Fomenko, A.T.** [4, p 94]), that is for $k \in Z$

$$H_k(M;\ Z) \simeq (\underbrace{Z \oplus \ldots \oplus Z}_{\beta_k \text{ times}}) \oplus (Z_{n_{k_1}} \oplus \cdots \oplus Z_{n_{kb(k)}}) \tag{3}$$

where $\beta_k = \beta_k(M;\ Z)$ are the Betti numbers of M with respect to the group $(Z, +)$, i.e. $\beta_k(M;\ Z) = \text{rank}\ H_k(M;\ Z)$, $k \in Z$.

Consider $H_k(M;Z)$, $k = \overline{0, m}$, the singular homology groups with the coefficients in the field F and $\beta_k(M;\ F) = \text{rank}\ H_k(M;\ F) = \dim_F H_k(M;\ F)$, $k = \overline{0, m}$, the Betti numbers of M with respect to F.

Put

$$\beta\ (M;\ Z) = \sum_{k=0}^{m} \beta_k(M;\ Z),\ \beta\ (M;\ F) = \sum_{k=0}^{m} \beta_k(M;\ F) \tag{4}$$

For $f \in \mathcal{F}_m(M)$ the following important relations hold:

$\mu_k(f) \geq \beta_k(M;\ F)$, $k = o, m$ **(weak Morse inequalities)**

$\sum\limits_{k=0}^{m} (-1)_k\ \mu_k(f) = \chi\ (M)$ **(Euler formula)**

(see **Andrica, D.** [1, Chapter 3] for the proof and interesting applications). Recall that, in the last relation, χ (M) represents the Euler–Poincare characteristic of M, i.e.

$$\chi\ (M) = \sum_{k=0}^{m} (-1)^k \dim_{\mathbb{R}} H^k(M), \tag{5}$$

where $H^k(M)$, $k = \overline{0, m}$, are the de Rham real cohomology spaces of M.

The Morse function $f \in \mathcal{F}_m(M)$ is **F-perfect** if the weak Morse inequalities become equalities, i.e.

$$\mu_k(f) = \beta_k(M;\ F),\ k = \overline{0, m}. \tag{6}$$

In the sequel we are interested in the following problem, which naturally appears in the theory of the tight and taut immersions (see, for instance, the excellent book of **Cecil, T.E., Ryan, P.J.** [3]):

Problem. For a given field F, characterize the manifolds which admit F–perfect Morse functions.

Concerning this question it is known the following result (see **Andrica, D.,** [1, Chapter 4], [2, Theorem 2]);

Theorem 1. The manifold M has F–perfect Morse functions if and anly if

$$\gamma(M) = \beta(M; F). \tag{7}$$

Let $p \geq 2$ be a prime number. Taking into account the relations (3) we can define

$$d(M, p) = \text{card } \{n_{kj},\ j = \overline{1, b(k)},\ k = \overline{0, m};\ p \mid n^k_j\}$$

The following result represents a necesary and sufficient condition, in terms of $\gamma(M)$, $\beta(M; Z)$ and $d(M, p)$, in order that the manifold M has Z_P–perfect Morse functions (see **Andrica, D.** [2, Theorem 4], [1, Chapter 4]).

Theorem 2. The manifold M has Z_P – perfect Morse functions an only if the following equality holds

$$\gamma(M) = \beta(M; Z) + 2d(M, p). \tag{8}$$

The main results. The aim of this note consists in an answer to the above problem when the manifold M^m is a smooth compact, connected, surface, i.e. the dimension of M is $m = 2$.

Let T^2 be the 2–dimensional torus, and let us define the smooth, compact, connected, orientable surface of the genus $g \geq 0$, by

$$T_g = \underbrace{T^2 \neq T^2 \neq \ldots \neq T^2}_{g \text{ times}}, \tag{9}$$

i.e. T_g is the connected sum of g copies of T^2. If $g = 0$, one considers $T_g = S^2$, the 2–dimensional sphere.

Consider P_g the smooth, compact, connected, and non– orientable surface, of genus $g \geq 0$, defined by

$$P_g = \underbrace{P\,\mathbb{R}^2 \neq P\,\mathbb{R}^2 \neq \ldots \neq P\,\mathbb{R}^2}_{(g+1) \text{ times}}, \tag{10}$$

where $P\,\mathbb{R}^2$ is the real projective plane.

It is well–known (see **Gramain, A.** [7]) that, if M is a smooth, compact, connected surface, without boundary, then M is diffeomorphic to T_g if it is orientable, and M is diffeomorphic to P_g if it is non– orientable, for some values of g.

The following result is a direct consequence of the well–known exact Mayer–Vietoris sequence in the de Rahm cohomology (see **Godbillon, C.**, [6, Proposition 1.3., p 179]):

$$\chi(T_g) = 2-2g, \quad \chi(P_g) = 1-g. \tag{11}$$

Kuiper, N.H. [8] (see also [9, § 4] or the book of **Cecil, T.E., Ryan, P.J.** [3, Proposition 5.6., p 26]) proved the following very interesting connection between the Morse–Smale characteristic given by (2) and the Euler–Poincare characteristic defined by (5), of a smooth, compact, connected surface M, without boundary:

$$\gamma(M) = 4 - \chi(M). \tag{12}$$

Using this result and the relations (11) one obtains

$$\gamma(T_g) = 2 + 2g, \quad \gamma(P_g) = 3 + g \tag{13}$$

Theorem 3.

(i) T_g has Q–perfect Morse functions.

(ii) For any prime number $p \geq 2$, T_g has Z_p– perfect Morse functions.

Proof: It is well–known (see **Lehman, D.,** Sacré, C. [10, Chapter IV, p 252–302]) that the integer homology of T_g is given by

$$H_k(T_g; Z) \simeq \begin{cases} Z \text{ if } k = 0 \\ \underbrace{Z \oplus \dots \oplus Z}_{2g \text{ times}} \text{ if } k = 1 \\ Z \text{ if } k = 2 \\ \{0\} \text{ otherwise.} \end{cases}$$

One obtains $\beta_0(T_g; Z) = \beta_2(T_g; Z) = 1$, $\beta_1(T_g; Z) = 2g$, and $d(T_g, p) = 0$ for any prime number $p \geq 2$.

(i) It is known (see Andrica, D. [2. Lemma 3], [1, Chapter 4]) that $\beta_k(M; Z) = \beta_k(M; Q)$, $k = o, m$; thus $\beta(T_g; Q) = 2 + 2g$. Taking into acoount the first relation in (13) it follows $\gamma(T_g) = \beta(T_g; Q)$, and the desired conclusion follows from Theorem 1.

(ii) Because $d(T_g, p) = 0$, for any prime number $p \geq 2$, one obtains $\gamma(T_g) = \beta(T_g; Z) + 2d(T_g, p)$, and the conclusion follows via Theorem 2.

Theorem 4.

(i) P_g has not Q–perfect Morse functions.

(ii) For any prime number $p \geq 3$, P_g has not Z_P – perfect Morse functions.

(iii) P_g has Z_2 – perfect Morse functions.

Proof: The singular homology of P_g (see the book of **Lehman, D., Sacre, C.** [10, Chapter IV, p 252–302]) is

$$H_k(P_g; Z) \simeq \begin{cases} Z \text{ if } k = 0 \\ Z_2 \oplus \underbrace{(Z \oplus \dots \oplus Z)}_{g \text{ times}} \text{ if } k = 1 \\ \{0\} \text{ otherwise} \end{cases}$$

Therefore $\beta_0(P_g; Z) = 1$, $\beta_1(P_g; Z) = g$, $\beta_2(P_g; Z) = 0$, and

$$d(P_g, p) = \begin{cases} 1 & \text{if} \quad p = 2 \\ 0 & \text{if} \quad p \geq 3. \end{cases}$$

One obtains $\beta(P_g; Z) = 1 + g = \beta(P_g; Q)$. Using the second relation in (13), it follows $\gamma(P_g) = 3 + g \neq \beta(P_g; Q)$, i.z. P_g has not Q – perfect Morse functions. Moreover,

$\gamma(P_g) = 3 + g = 1 + g + 2 = \beta(P_g; Z) + 2d(P_g; 2)$, i.e. P_g admits Z_2 – perfect Morse functions. In an analogous way one can obtain the conclusion (ii).

Remark. For $g = 0$ the above results appear in the author's paper [2, Theorems 7, 8] for the m–dimensional manifolds S^m and $P\,\mathbb{R}^m$.

REFERENCES

[1] ANDRICA, D., Critical Point Theory and Some Applications, University of Ankara, (1992), 150 pp.

[2] ANDRICA, D., On the F–perfect Morse functions on compact manifolds, Southeast Asian Bull. of Mathematics (to appear), 12 pp.

[3] CECIL, T.E., RYAN, P.J., Tight and taut immersions of manifolds, Research Notes in Math., Vol. 107, Pitman, 1985.

[4] FOMENKO, A.T., a.o., Homotopic Topology, Akademiai Kiado, Budapest, 1986.

[5] GODBILLON, C., Elements de topologie algébrique, Hermann, Paris, 1971.

[6] GODBILLON, C., Dynamical Systems on Surfaces, Springer–Verlag, Berlin, Heidelberg, New York, 1983.

[7] GRAMAIN, A., Topologie des Surfaces, Presses Univ. de France, Paris, 1971.

[8] KUIPER, N.H., On surfaces in Euclidean three space, Bull. Soc. Math. Belg. 12 (1960), 5–12.

[9] KUIPER, N.H., Tight embeddings and maps. Submanifolds of geometrical class three in E^n, The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp 79–145, Springer–Verlag, Berlin, Heidelberg, New York, 1980.

[10] LEHMAN, D., SACRE, C., Geometrie et topologie de surgaces, Presses, Univ. de France, Paris, 1982.