Commun. Fac. Sci. Univ. Ank. Series A_1
V. 42. pp. 61-67 (1993)

A FIXED POINT THEOREM ON BANACH SPACE AND ITS APPLICATIONS

H.K. PATHAK

Department of Mathematics Kalyan Mahavidyalaya Bhilai Nagar (M.P.), 49006 *INDIA*
(Received May 20, 1993; Accepted Sep. 17, 1993)

ABSTRACT

In Section 1 of this paper we have extended the results on fixed point of operators on Banach spaces of Bernfeld, Lakshmikantham and Reddy and Som for two set-valued mappings. In the proof of our theorems we have introduced a generalized version of a contrative type condition. In Section 2, we have investigated the solvability of certain non-linear functional equations in Banach spaces.

1. INTRODUCTION

Let X be a Banach space and let B(X) be the set of all non-empty, bounded subsets of X. Let C denotes the Banach space of all continuous functions from a finite closed interval [a, b] into X, such that

$$\|f\|_C = \sup_{a \leq t \leq b} \|f(t)\|_X \text{ for all } f \in C.$$

The function $\|A-B\|_{B(X)}$ with A,B in B(X) is defined by

$$\|A-B\|_{B(X)} = \sup \{\|a-b\|_X : a \in A, b \in B\}.$$

If A consists of a single point a we write

$$\|A-B\|_{B(X)} = \|a-B\|_{B(X)}$$

and if B also consists of a single point b we write

$$\|A-B\|_{B(X)} = \|a-b\|_{B(X)} = \|a-b\|_X.$$

If follows easily from the definition that

$$\|A-B\|_{B(X)} = \|B-A\|_{B(X)} \geq 0$$

and

$\|A-B\|_{B(X)} \leq \|A-C\|_{B(X)} + \|C-B\|_{B(X)}$, see Fisher [2,3] and Kaulgud and Pai [4].

Now, let F be a mapping of C into B(X), a member $f \in C$ is said to be a fixed point of F if F(e) is in Ff for some $e \in [a, b]$.

The intent of the present paper is to extend a result of Bernfeld et al. [4] for two set - valued mappings F and G of C into B(X), where C is a Banach space of all continuous functions from a finite closed interval [a,b] into X and B(X) being the set of all nonempty, bounded subsets of a Banach space X. One may also observed that we have introduced a generalized version of a contractive type condition.

Now, we give our main results as follows.

Theorem 1.1. Let $F : C \to B(X)$ and $G : C \to B(X)$ be two mappings which satisfy the following conditions for all $f, g \in C$, for a given $e \in [a,b]$

(a) $\|Ff—Gg\|_{B(X)} \leq \{\|f(e)—Gg\|_{B(X)}\}^{1-\gamma-\delta}\{\|g(e)—Ff\|_{B(X)}\}^{\gamma}\{\|f—g\|_{C}\}^{\delta}$ where $0 \leq q \leq 1$, $\gamma, \delta \in [0, 1]$ with $0 < \gamma + \delta < 1$. Then (i) for a given $f_c \in C$, every sequence $\{f_n : n=1,2,\ldots\}$ defined as $f_n(e)$ is in $Ff_{n-1} = X_n$ for the some $e \in [a,b]$ such that

$$\|f_{n1} — f_{n2}\|_C = \|f_{n1}(e) — f_{n2}(e)\|_X$$

for $n_1, n_2 = 0,1,2,\ldots$, converges to a point f^* and f^* is a common fixed point of F and G. Further, $Ff^* = Gf^* = \{f^*(e)\}$ and $f^*(e)$ is the unique common point in Ff^* and Gf^*, and

(ii) Let $\Omega f^* = \{f \in C : \|f—f^*\|_C = \|f^*(e)\|_X\}$,

where f^* is a common fixed of F and G, then F and G have a unique common fixed point in Ωf^*.

Proof: First we prove that $\{f_n\}$ is a Cauchy sequence. Put

$$\alpha = \|f_o(e) — Gf_o\|_{BX)}$$

and suppose that the sequence $\{\|X_n—Gf_o\|_{B(X)} : n = 1,2,\ldots\}$ is unbounded. Then there exists an integer $n \geq 2$ such that

$$\beta = \|x_n — Gf_o\|_{B(X)} \geq \|X_{n-1} — Gf_o\|_{B(X)}$$

with $\beta > q^{1/(\gamma+\delta)}\alpha/(1—q^{1/(\gamma+\delta)})$ and so

$$\|X\gamma—f_o(e)\|_{B(X)} \leq \|X\gamma—Gf_o\|_{B(X)} + \|Gf_o—f_o(e)\|_{B(X)} \leq \beta+\alpha$$

for $r = n—1$, n. But on using inequality (a), we have

$$\beta = \|X_n—Gf_o\|_{B(X)} = \|Ff_{n-1}—Gf_o\|_{B(X)}$$

$$\leq q\{\|f_{n-1}(e)—Gf_o\|_{B(X)}\}^{1-\gamma-\delta}\{\|f_o(e)—Ff_{n-1}\|_{B(X)}\}^{\gamma}\{\|f_{n-1}—f_o\|_C\}^{\delta}$$

$\leq q\{\|X_{n-1}-Gf_0\|_{B(X)}\}^{1-\gamma-\delta}\{\|f_0(e)-X_n\|_{B(X)}\}^{\gamma}\{\|f_{n-1}(e)-f_0(e)\|_X\}^{\delta}$

$\leq q\{\|X_{n-1}-Gf_0\|_{B(X)}\}^{1-\gamma-\delta}\{\|f_0(e)-X_n\|_{B(X)}\}^{\gamma}\{\|X_{n-1}-f_0(e)\|_{B(X)}\}^{\delta}$

$\leq q\ \beta^{1-\gamma-\delta}(\beta+\alpha)^{\gamma}\ (\beta+\alpha)^{\delta}$

which implies that

$$\beta < q^{1/(\gamma+\delta)}\ \alpha/(1-q^{1/(\gamma+\delta)})$$

giving a contradiction. The sequence $\{\|X_n-Gf_0\|_{B(X)} : n=1,2,\ldots\}$ must therefore be bounded.

Similarly, we can prove that the sequence $\{\|Fg_0-Y_n\|_{B(X)} : n=1, 2,\ldots\}$ is bounded. Since

$\|X\gamma-Y_s\|_{B(X)} \leq \|X\gamma-Gf_0\|_{B(X)} + \|Gf_0-Fg_0\|_{B(X)} + \|Fg_0-Y_s\|_{B(X)}$,
it follows that

$$M = \sup\ \{\|X\gamma-Y_s\|_{B(X)} : r,\ s, = 1,2,\ldots\}$$

is finite. Now, for arbitrary $\varepsilon > 0$, chose o a positive integer N such that

$$q^N\ M < \varepsilon$$

Then for m, n > N

$\|X_m-Y_n\|_{B(X)} = \|Ff_{m-1}-Gg_{n-1}\|_{B(X)}$

$\leq q\{\|f_{m-1}(e)-Gg_{n-1}\|_{B(X)}\}^{1-\gamma-\delta}\{\|g_{n-1}(e)-Ff_{m-1}\|_{B(X)}\}^{\gamma}\{\|f_{m-1}-g_{n-1}\|\}^{\delta}$

$\leq q\{\|X_{m-1}-Y_n\|_{B(X)}\}^{1-\gamma-\delta}\{\|Y_{n-1}-X_m\|_{B(X)}\}^{\gamma}\{\|f_{m-1}(e)-g_{n-1}(e)\|_X\}^{\delta}$

$\leq q\{\|X_{m-1}-Y_n\|_{B(X)}\}^{1-\gamma-\delta}\{\|Y_{n-1}-X_m\|_{B(X)}\}^{\gamma}\{\|X_{m-1}-Y_{n-1}\|_{B(X)}\}^{\delta}$

$\leq q\{\|X\gamma-Y_s\|_{B(X)} : m-1 \leq \gamma \leq m;\ n-1 \leq s \leq n.\}$

$\leq q^2\{\|X\gamma-Y_s\|_{B(X)} : m-2 \leq \gamma \leq m;\ n-2 \leq s \leq n\}$

....

....

$\leq q^n\{\|X\gamma-Y_s\|_{B(X)} : m-N \leq \gamma \leq m;\ n-N \leq s \leq n\}$

$\leq q^n\ M < \varepsilon.$

Thus

$$\begin{aligned}\|f_m-f_n\|_C &= \|f_m(e)-f_n(e)\|_X\\ &\leq \|X_m-X_n\|_{B(X)}\\ &\leq \|X_m-Y_n\|_{B(X)} + \|Y_n-X_n\|_{B(X)} \leq 2\ \varepsilon\end{aligned}$$

for m, n > N. The sequence $\{f_n : n=1,2,\ldots\}$ is therefore a Cauchy sequence in Banach space C and so has a limit f*. Further,

$$\|f_m(e)-X_n\|_{B(X)} \leq \|X_m-X_n\|_{B(X)} \leq 2\varepsilon$$

for m, n > N and so on letting m tends to infinity we have

$$\|f^*(e)-X_n\|_{B(X)} \leq 2\varepsilon$$

for n > N. Thus

$$\|f_n(e)-Gf^*\|_{B(X)} \leq \|Ff_{n-1}-Gf^*\|_{B(X)}$$

$$\leq q\{\|f_{n-1}(e)-Gf^*\|_{B(X)}\}^{1-\gamma-\delta}\{\|f^*(e)-X_n\|_{B(X)}\}^{\gamma}\{\|f_{n-1}.f^*\|_C\}^{\delta}$$

$$\leq q\{\|f_{n-1}(e)-Gf^*\|_{B(X)}\}^{1-\gamma-\delta}(2\varepsilon)^{\gamma}\{\|f_{n-1}-f^*\|_C\}^{\delta}$$

for n > N. Letting n tends to infinity, we have

$$\|f^*(e)-Gf^*\|_{B(X)} \leq q\{\|f^*(e)-Gf^*\|_{B(X)}\}^{1-\gamma-\delta}(2\varepsilon)^{\gamma}\{\|f^*-f^*\|_C\}^{\delta}$$

which implies that

$$\|f^*(e)-Gf^*\|_{B(X)} = 0$$

and so

$$Gf^* = \{f^*(e)\}$$

We now have

$$\|Ff^*-f^*(e)\|_{B(X)} \leq \|Ff^*-Gf^*\|_{B(X)}$$

$$\leq q\{\|f^*(e)-Gf^*\|_{B(X)}\}^{1-\gamma-\delta}\|f^*(e)-Ff^*\|_{B(X)}\}^{\gamma}\{\|f^*-f^*\|_C\}^{\delta}$$

$$= 0.$$

It follows that $Ff^* = \{f^*(e)\}$ and f^* is a common fixed point of F and G and $f^*(e)$ is a common point in Ff^* and Gf^*.

Now, suppose that G has a second fixed point f^{**} so that $f^{**}(e)$ is in Gf^{**}. Then

$$\|f^*(e) - f^{**}(e)\|_X \leq \|f^*(e) - Gf^{**}\|_{B(X)}$$

$$\leq \|Ff^*-Gf^{**}\|_{B(X)}$$

$$\leq q\{\|f^*(e)-Gf^{**}\|_{B(X)}\}^{1-\gamma-\delta}\{\|f^{**}(e)-Ff^*\|_{B(X)}{}^{\gamma}\{\|f^*-f^{**}\|_C\}^{\delta}$$

$$\leq q\{\|Ff^*-Gf^{**}\|_{B(X)}\}^{1-\gamma-\delta}\{\|Gf^{**}-Ff^*\|_{B(X)}\}^{\gamma}\{\|Ff^*-Gf^{**}\|_{B(X)}\}^{\delta}$$

which implies

$$\|Ff^*-Gf^{**}\|_{B(X)} \leq q\|Ff^*-Gf^{**}\|_{B(X)}$$

a contraction, since $q < 1$. It follows that

$$\|Ff^*-Gf^{**}\|_{B(X)} = 0.$$

Thus, we have

$$\|f^*(e)-Gf^{**}\|_{B(X)} = 0$$

and then that $f^*(e) = f^{**}(e)$. The set Gf^* is therefore singleton and $f^*(e)$ is in Gf^*. Similarly, the set Ff^* is singleton and $f^*(e)$ is in Ff^*.

Finally, let $g^*(\neq f^*)$ be another common fixed point of F and G in Ωf^* where f^* is a common fixed point of F and G in Ωf^*, then

$$\|f^*-g^*\|_C = \|f^*(e)-g^*(e)\|_{B(X)} = \|Ff^*-Gf^*\|_{B(X)}$$
$$\leq q\{\|f^*(e)-Gf^*\|_{B(X)}\}^{1-r-\delta}\{\|g^{**}(e)-Ff^*\|_{B(X)}\}^r\{\|f^*-g^*\|_C\}^\delta$$
$$= q\{\|f^*(e)-g^*(e)\|_{B(X)}\}^{1-r-\delta}\{\|g^*(e)-f^*\|_{B(X)}\}\{\|f^*-g^*\|_C\}^\delta$$
$$= q\{\|f^*-g^*\|_C\}^{1-r-\delta}\{\|g^*-f^*\|_C\}^r\{f^*-g^*\|_C\}^\delta$$
$$= q\|f^*-g^*\|_C$$

a contradiction, since $q < 1$. It follows that

$$\|f^*-g^*\|_C = 0.$$

Hence $f^* = g^*$, that is, F and G have a unique common fixed point f^* in Ωf^*.

Corollary 1. Let $S : C \to X$ and $T : C \to X$ be two operators satisfying the following condition for all $f, g \in C$, and for a given $e \in [a,b]$

$$\|Sf-Tg\|_X \leq q\{\|f(e)-Tg\|_X\}^{1-\gamma-\delta}\{\|g(e)-Sf\|_X\}^\gamma\{\|f-g\|_C\}^\delta$$

where $0 \leq q \leq 1$, $\gamma, \delta \in [0,1]$ with $0 \leq \gamma + \delta \leq 1$. Then

(i) for a given $f_0 \in C$, every sequence $\{f_n : n=1,2,\ldots\}$ defined as $f_n(e) = Sf_{n-1}$ for the same e such that

$$\|f_{n1}-f_{n2}\|_C = \|f_{n1}(e)-f_{n2}(e)\|_X$$

for $n_1, n_2 = 0,1,2,\ldots$, converges to a point f^* and f^* is a common fixed point of S and T, and

(ii) Let $\Omega f^* = \{f \in C: \|f-f^*\|_C = \|f(e)-f^*(e)\|_X\}$, where f^* is a common fixed point of S and T, then S and T have a unique fixed point f^* in Ωf^*.

Proof: Define mappings F and G of C into B(X) by putting

$$Ff = \{Sf\}, \quad Gf = \{Tg\}$$

for all $f, g \in C$ It follows that F and G satify the conditions of Theorem 1 and so there exists a point f^* in C with

$$Ff^* = Gf^* = \{f^*(e)\}.$$

Hence, f^* is a common fixed point of S and T. Further, it is easy to see that $f^*(e)$ is the unique point in Ff^* and Gf^*. The uniqueness of f^* is immediate.

The following corollary is an immediate consequence of Corollary 1.

Corollary 2. (Som [5]) Let $T: C \to X$ be an operator which satisfies the following condition for all $f, g \in C$ for a given $e \in [a, b]$

$$\|Tf - Tg\|_X \leq q \{\|f(e) - Tg\|_X\}^{1-\gamma-\delta} \{\|g(e) - Tf\|_X\}^{\gamma} \{\|f - g\|_C\}^{\delta}$$

where $0 \leq q < 1$, $\gamma, \delta \in [0, 1]$ with $0 \leq \gamma + \delta \leq 1$. Then (i) for a given $f_0 \in C$, every sequence $\{f_n = n\}$ 1,2,...} defined as $f_n(e) = Ff_{n-1}$ for the same e each that

$$\|f_{n1} - f_{n2}\|_C = \|f_{n1}(e) - f_{n2}\|_X$$

for $n_1, n_2 = 0, 1, 2, \ldots$, converges to a fixed point f^* of T, and

(ii) Let $\Omega f^* = \{f \in C: \|f - f^*\|_C = \|f(e) - f^*(e)\|_X\}$, where f^* is a fixed point of T, then T has a unique fixed point f^* in Ωf^*.

Remark: If we take $\gamma = 0$ and $\delta = 1$ in the above corollary, the results of Bernfeld et al. [1] follows.

2. **Application:** In this section we wish to investigate the solvability of certain non-linear functional equations in a Banach space.

Theorem 2.1. Let $\{f_n\}$ be a sequence of elements in C, and let $\{g_n\}$ be a sequence of solutions to the equation $\|Gf - Ff\|_{B(X)} = \|f_n(e)\|_X$ for the same $e \in [a,b]$, $n = 1,2,\ldots$, where F and G are as in Theorem 1.1, $f_n(e)$ is in Ff_{n-1} and $g_n(e)$ is in Fg_{n-1}. Then if $\|f_n(e)\|_X \to 0$ is $n \to \infty$, the sequence $\{g_n\}$ converges to the unique solution of the equation $Ff = Gf$.

Proof: First we observe that

$$\|Fg_n - Gg_m\|_{B(X)} \leq q\{\|g_n(e) - Gg_m\|_{B(X)}\}^{1-\gamma-\delta}\{\|g_m(e) - Fg_n\|_{B(X)}\}^{\gamma} \{\|g_n - g_m\|_C\}^{\delta}$$

$$\leq q\{\|Fg_n - Gf_m\|_{B(X)}\}^{1-\gamma-\delta}\{\|Fg_m - Gf_n\|_{B(X)}\}^{\gamma}\{\|g_n(e) - g_m(e)\|_{B(X)}\}^{\delta}$$

$$\leq q\{\|Fg_n - Gg_m\|_{B(X)}\}^{1-\gamma-\delta}\{\|Fg_m - Ff_n\|_{B(X)}\}^{\gamma}\{\|Fg_n - Fg_m\|_{B(X)}\}^{\delta}$$

$$< q\{\|Fg_n - Gg_m\|_{B(X)}\}^{1-\gamma-\delta}\{\|Fg_m - Gg_m\|_{B(X)} + \|Fg_n - Gg_m\|_{B(X)}\}^{r+\delta}$$

which implies

$$\{\|Fg_n - Gg_m\|_{B(X)}\}^{\gamma+\delta} \leq q\{\|f_m(e)\|_X + \|Fg_n - Gg_m\|_{B(X)}\}^{\gamma+\delta}$$

So we have

$$\|Fg_n - Gg_m\|_{B(X)} \leq \frac{q^{1/(\gamma+\delta)} \|f_m(e)\|_X}{1 - q^{1/(\gamma+\delta)}}.$$

Now, clearly

$$\|g_n - g_m\|_C = \leq \|g_n(e) - g_m(e)\|_{B(X)} \leq \|Fg_n - Fg_m\|_{B(X)}$$
$$\leq \|Fg_n - Gg_m\|_{B(X)} + \|Fg_m - Gg_m\|_{B(X)}$$
$$\leq \frac{q^{1/(\gamma+\delta)} \|f_m(e)\|_X}{1 - q^{1/(\gamma+\delta)}} + \|f_m(e)\|_X \ .$$

Hence $\{g_n\}$ is a Cauchy sequence and so it will converge to some point, say g^*. We further note from the inequality (a)

$$\|g_n(e) - Gg^*\|_{B(X)} \leq \|Fg_n - Gg^*\|_{B(X)}$$
$$\leq q \{ \|g_n(e) - Gg\|_{B(X)} \}^{1-\gamma-\delta} \{ \|g^*(e) - Fg_n\|_{B(X)} \}^{\gamma} \{ \|g_n - g^*\|_C \}^{\delta}.$$

Letting $n \to \infty$, we have

$$\|g^*(e) - Gg^*\| = 0$$

and so

$$Gg^* = g^*(e).$$

We now have

$$\|Fg^* - g^*(e)\|_{B(X)} \leq \|Fg^* - Gg^*\|_{B(X)}$$
$$\leq q \|g^*(e) - Gg^*\|_{B(X)} \}^{1-\gamma-\delta} \{ \|g^*(e) - Fg^*\|_{B(X)} \}^{\gamma} \{ \|g^* - g^*\|_C \}^{\delta} = 0.$$

It follows that $Fg^* = \{g^*(e)\}$ and $\{g_n\}$ converges to the solution of the equation $Ff = Gf$ as required. The rest of the proof is simple.

Acknowledgement

It is a great pleasure to thank Prof. Brain Fisher for sending me reprints of some of his work.

REFERENCES

[1] BERNFELD, S. LAKSHMIKANTHAM, V., & REDDY, V.M., Fixed point of operators on Banach spaces, J. Appl. Anal. 6 (1977), 271–280.

[2] FISHER, B., Set-valued mappings, on metric spaces Fund. Math. 112 (1981), 141–45.

[3] FISHER, B., Fixed point of mappings and set-valued mappings, J. Univ. Kuwait. (Sci) 9 (1982), 175–79.

[4] KAULGUD, N.N., & PAI, D.V., Fixed point theorems for set-valued mapping, Nieuw Arch. Wisk. 23 (1975), 49–66.

[5] SOM, T., Some fixed point theorems of metric and Banach spaces, Indian J. Pure Appl. Math. 16 (6) (1985), 575–585.