

ON THE CONGRUENCES OF LINES GENERATED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH SOME SURFACES

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In this paper, the surfaces \vec{z} and \vec{z} in [2] have been considered as the base surface \vec{x} . And they are referred to their lines of curvature. The congruences generated by the instantaneous screwing axes of the moving trihedrons, moving along the lines of curvature on \vec{z} and \vec{z} , have been investigated. And some corresponds with [1] have been explained.

1. INTRODUCTION

1. Let a surface \vec{x} be referred to its lines of curvature. In case the congruences \vec{y} and \vec{y} , generated by the instantaneous screwing axes \vec{G} and \vec{G} of the moving trihedrons connected with these lines are normal congruences, the surfaces which generate them are consecutively,

$$\vec{z} = \vec{r} + \frac{1}{b} \vec{g} \tag{1.1}$$

$$\vec{r} = \vec{x} + \frac{1}{r} \vec{\xi} \quad \vec{\xi} = \vec{z} + \frac{1}{b} \vec{n}, \quad \vec{g} = \frac{r\vec{x}_2 + q\vec{\xi}}{\sqrt{r^2 + q^2}}$$

and

$$\vec{z} = \vec{r} + \frac{1}{\beta} \vec{g} \tag{1.2}$$

$$\vec{r} = \vec{x} + \frac{1}{\bar{r}} \vec{\xi} \quad \vec{\xi} = \vec{z} + \frac{1}{\beta} \vec{n}, \quad \vec{g} = \frac{r\vec{x}_1 + q\vec{\xi}}{\sqrt{\bar{r}^2 + q^2}}$$

([1] and [2]).

2. The Surfaces \vec{z} , \vec{z} as base surfaces.

1) The derivative formulas of the moving trihedron $(\vec{z}_2, -\vec{z}_1, \vec{n})$ of the lines of curvature $u = \text{const.}$ with tangent \vec{z}_2 on the surface z , may be stated as

$$\begin{aligned}\vec{z}_{22} &= -a\vec{z}_1 + b\vec{n} \\ \vec{z}_{12} &= a\vec{z}_2 \\ \vec{n}_2 &= -b\vec{z}_2\end{aligned}\quad (2.1)$$

Here, $\tau_{gv} = (\vec{z}_2, \vec{n}, \vec{n}_2) = 0$, $\bar{a} = (\vec{n}, \vec{n}_2, \vec{z}_{22})$, $\bar{b} = -(\vec{z}_2, \vec{n}_2)$.

And the derivative formulas of the moving trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$ of the lines of curvature $v = \text{const.}$ on the surface \vec{z} are

$$\begin{aligned}\vec{z}_{11} &= -a\vec{z}_2 + b\vec{n} \\ \vec{z}_{21} &= a\vec{z}_1 \\ \vec{n}_1 &= -b\vec{z}_2\end{aligned}\quad (2.1)$$

Here $\tau_{gu} = (\vec{z}_1, \vec{n}, \vec{n}_1) = 0$, $-a = (\vec{n}_1, \vec{z}_{11})$, $b = -(\vec{z}_1, \vec{n}_1)$.

The surface \vec{z} being considered as a base surface and during its motion on the lines of curvature $u = \text{const.}$ if the axes of the moving trihedron $(\vec{z}_2, -\vec{z}_1, \vec{n})$ are denoted by $\vec{Z}_1 = \vec{z}_1 + \varepsilon z_{10}$, $\vec{Z}_2 = \vec{z}_2 + \varepsilon z_{20}$, $\vec{Z}_3 = \vec{n} + \varepsilon n_0$, ($\varepsilon^2 = 0$), its instantaneous screwing axis \vec{G}^* becomes,

$$\vec{G}^* = \frac{\varepsilon \vec{Z}_2 + \vec{bZ}_1 + \vec{aZ}_3}{\sqrt{\vec{a}^2 + \vec{b}^2}}, \quad (\vec{a} \neq 0, \vec{b} \neq 0). \quad (2.2)$$

Separating \vec{G}^* into its real and dual parts, we get

$$\vec{G}^* = \frac{\vec{bz}_1 - \vec{an}}{\sqrt{\vec{a}^2 + \vec{b}^2}} + \varepsilon \frac{\vec{z}_2 + \vec{bz}_{10} + \vec{a} \vec{n}_0}{\sqrt{\vec{a}^2 + \vec{b}^2}} = \vec{g}^* + \varepsilon \vec{g}_0 \quad (2.2)$$

$$\vec{g}^* = \frac{\vec{bz}_1 + \vec{an}}{\sqrt{\vec{a}^2 + \vec{b}^2}} \quad \vec{g}_0^* = \frac{\vec{z}_2 + \vec{bz}_{10} + \vec{a} \vec{n}_0}{\sqrt{\vec{a}^2 + \vec{b}^2}}.$$

From (2.1) and from the above definitions of \vec{a} , \vec{b} we find,

$$\vec{b} = \frac{1}{\lambda} \quad \left(\lambda = \frac{1}{\vec{b}} \right) \quad (2.3)$$

$$\vec{a} = \frac{q}{\lambda r} \quad \left(\frac{\vec{a}}{\vec{b}} = \frac{q}{r} \right).$$

If we substitute these into \vec{g}^* , we find

$$\vec{g}^* = \vec{\xi} \quad (2.4)$$

From this the following theorem may be stated:

2.1. Theorem

The instantaneous screwing axis \vec{G}^* of the trihedron $(\vec{z}_2, -\vec{z}_1, \vec{n})$, moving along the lines of curvature $u = \text{const.}$ on the base surface \vec{z} , coincides with the axis \vec{X}_3 carrying the surface normal $\vec{\xi}$ of the trihedron $(\vec{x}_1, \vec{x}_2, \vec{\xi})$ moving along the lines of curvature $v = \text{const.}$ on the base surface \vec{x} .

The following theorem may be derived from (2.3).

2.2. Theorem

During the motion considered in Theorem 2.1, the ratio of the geodesic curvatures and the normal curvatures of the surface \vec{x} and \vec{z} , are equal $\left(\frac{q}{r} = \frac{\bar{a}}{b}\right)$. In other words, the angles between the surface normals and the principal normals of these surfaces are equal ($\text{tg}\theta = \text{tg}\gamma$).

2) As the derivarative formulas of the trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$ of the lines of curvature $v = \text{const}$. On the surface \vec{z} , we may write.

$$\begin{aligned}\frac{\vec{z}}{1} \vec{z}_1 &= -\alpha \vec{z}_2 + \beta \vec{n} \\ \frac{\vec{z}}{2} \vec{z}_1 &= \alpha \vec{z}_1 \\ \frac{\vec{z}}{1} \vec{n} &= -\beta \vec{z}_1\end{aligned}\tag{2.5}$$

Here, $\tau_{gu} = (\vec{z}_1 \vec{n} \vec{n}_1) = 0$, $-\alpha = (\vec{n} \vec{z}_1 \vec{z}_2)$, $\beta = -(\vec{z}_1 \vec{n}_1)$.

Also from (2.5) the derivative formulas of the trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$ of the lines of curvature $u = \text{const}$. on \vec{z} , are

$$\begin{aligned}\frac{\vec{z}}{2} \vec{z}_2 &= -\bar{\alpha} \vec{z}_1 + \beta \vec{n} \\ \frac{\vec{z}}{1} \vec{z}_2 &= \bar{\alpha} \vec{z}_2 \\ \frac{\vec{z}}{2} \vec{n} &= -\beta \vec{z}_2.\end{aligned}\tag{2.5}$$

Here, $\tau_{gv} = (\vec{z}_2 \vec{n} \vec{n}_2) = 0$, $\alpha = (\vec{n} \vec{z}_1 \vec{z}_2)$, $\beta = -(\vec{z}_2 \vec{n}_2)$.

The surface \vec{z} being considered as a base surface and during its motion on the lines of curvature $v = \text{const.}$, if the axes of the moving trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$ are denoted by $\vec{Z}_1 = \vec{z}_1 + \epsilon \vec{z}_{10}$, $\vec{Z}_2 = \vec{z}_2 + \epsilon \vec{z}_{20}$, $\vec{Z}_3 = \vec{n} + \epsilon \vec{n}_0$, its instantaneous screwing axis \vec{G}^{**} becomes

$$\vec{G}^{**} = \frac{\epsilon \vec{Z}_1 - \beta \vec{Z}_2 - \alpha \vec{Z}_3}{\sqrt{\alpha^2 + \beta^2}}, \quad (\alpha \neq 0, \beta \neq 0). \tag{2.6}$$

Separating \vec{G}^{**} into its real and dual parts, we get

$$\vec{G}^{**} = - \frac{\beta \vec{z}_2 + \alpha \vec{n}}{\sqrt{\alpha^2 + \beta^2}} + \epsilon \frac{\vec{z}_1 \quad \beta \vec{z}_{20} \quad \alpha \vec{n}_0}{\sqrt{\alpha^2 + \beta^2}} = \vec{g}^{**} + \epsilon \vec{g}_0^{**} \tag{2.6}'$$

$$\vec{g}^{**} = - \frac{\beta \vec{z}_2 + \alpha \vec{n}}{\sqrt{\alpha^2 + \beta^2}}, \quad \vec{g}_0^{**} = \frac{\vec{z}_1 - \beta \vec{z}_{20} - \alpha \vec{n}_0}{\sqrt{\alpha^2 + \beta^2}}.$$

From (2.5) and from the above definitions of α and β , we find

$$\beta = \frac{1}{\lambda} \quad \left(\bar{\lambda} = \frac{1}{\beta} \right) \tag{2.7}$$

$$\alpha = - \frac{1}{\lambda r} \quad \left(\frac{\alpha}{\beta} = - \frac{\bar{q}}{\bar{r}} \right).$$

If we substitute these into \vec{g}^{**} , we find

$$\vec{g}^{**} = - \vec{\xi}. \tag{2.8}$$

Therefore, the following theorem may be stated:

2.3. Theorem

The instantaneous screwing axis \vec{G}^{**} of the trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$, moving along the lines of curvature $v = \text{const.}$ on the base surface \vec{z} coincides with $-\vec{X}_3$ axis carrying the surface normal $\vec{\xi}$ of the trihedron $(\vec{x}_2, -\vec{x}_1, \vec{\xi})$, moving along the lines of curvature $u = \text{const.}$ on the base surface \vec{x} .

The following theorem may be derived from (2.7):

2.4. Theorem

During the motion considered in Theorem 2.3, the ratio of the geodesic curvatures and the normal curvatures of the surfaces \vec{x} and \vec{z} , are equal with a minus sign $\left(\frac{\alpha}{\rho} = -\frac{\bar{q}}{\bar{r}}\right)$. In other words, the angles between the surface normals and the principal normals of these surfaces are equal with a minus sign ($\text{tg } \omega = -\text{tg } \bar{\theta}$).

3. The congruences generated by the instantaneous screwing axes connected with the moving trihedrons of the surfaces \vec{z} , \vec{z} .

1) Considering $z_{gV} = 0$, $\rho_{gV} = \bar{a}$, $\rho_{nV} = \bar{b}$, the instantaneous screwing axis of the moving trihedron $(\vec{z}_2, -\vec{z}_1, \vec{n})$, moving along the lines of curvature $u = \text{const.}$ on the surface \vec{z} (u, v) is seen in (2.2) and (2.2)'. Therefore, the congruence generated by \vec{G}^* may be expressed as

$$\vec{y}^* = \vec{r} + t^* \vec{g}^*, \quad (\vec{g}^* = 1). \quad (3.1)$$

Here, the reference surface \vec{r} .

$$\vec{r} = \vec{z} + \frac{1}{\bar{b}} \vec{n} \quad (3.2)$$

is the center surface belonging to the line curvature $u = \text{const.}$ on the surface \vec{z} [2].

Therefore, the vectorial equation of the congruence which will be investigated may be stated as

$$\begin{aligned} \vec{y}^* &= \vec{y}^*(u, v, \vec{t}^*) = \vec{z}(u, v) + \frac{1}{\bar{b}(u, v)} \vec{n}(u, v) + \\ &+ \vec{t}^* \frac{\bar{b}(u, v) \vec{z}_1(u, v) + \bar{a}(u, v) \vec{n}(u, v)}{\sqrt{\bar{a}^2(u, v) + \bar{b}^2(u, v)}} \end{aligned}$$

2. Considering $\tau_{gu} = 0$, $\rho_{gu} = -\alpha$, $\rho_{uu} = \beta$, the instantaneous screwing axis of the trihedron $(\vec{z}_1, \vec{z}_2, \vec{n})$, moving along the lines of curvature $v = \text{const.}$ on the surface $\vec{z}(u, v)$ is seen in (2.6) and (2.6)'. Therefore, the congruence generated by \vec{G}^{**} may be expressed as

$$\vec{y}^{**} = \vec{r} + \vec{t}^* \vec{g}^{**}, \quad (g^{**2} = 1). \tag{3.3}$$

Here, the reference surface \vec{r} ,

$$\vec{r} = \vec{z} + \frac{1}{\beta} \vec{n} \tag{3.4}$$

is the central surface belonging to the lines of curvature $v = \text{const.}$ on the surface \vec{z} [2].

The vectorial equation of the line congruence to be investigated may be written as

$$\begin{aligned} \vec{y}^{**} &= \vec{y}^{**}(u, v, \vec{t}^{**}) = \vec{z}(u, v) + \frac{1}{\beta(u, v)} \vec{n}(u, v) \\ &- \vec{t}^{**} \frac{\beta(u, v) \vec{z}_2(u, v) + \alpha(u, v) \vec{n}(u, v)}{\sqrt{\alpha^2(u, v) + \beta^2(u, v)}} \end{aligned}$$

4. Properties of the congruences \vec{y}^* , \vec{y}^{**} .

In order to investigate the properties of the congruences \vec{y}^* , \vec{y}^{**} before calculating the first and second fundamental forms, in the KUMMER sense, according to (2.4) and (2.8), (Theorems 2.1 and 2.3), the below theorem may be stated:

4.1. Theorem

The congruences \vec{y}^* and \vec{y}^{**} are normal congruences.

\vec{y}^* 1) The first and second fundamental forms of the congruence \vec{y}^* are calculated

$$\vec{g}^* = \vec{g}_u^{*2} = \vec{\xi}_u^2 = \vec{\xi}_1^2 \vec{E} = \vec{\xi}_1^2 E = r^2 E$$

$$\vec{g}^* = \vec{g}_u^{*2} \cdot \vec{z}_u^* = \vec{\xi}_u \cdot \vec{\xi}_v = \vec{\xi}_1 \cdot \vec{\xi}_2 \sqrt{\vec{E} \vec{G}} = \vec{\xi}_1 \cdot \vec{\xi}_2 \sqrt{EG} = 0 \quad (4.1.)$$

$$\vec{g}^* = \vec{g}_v^{*2} = \vec{\xi}_v^2 = \vec{\xi}_2^2 \vec{G} = \vec{\xi}_2^2 G = \bar{r}^2 G,$$

$$d\sigma^{*2} = r^2 Edu^2 + \bar{r}^2 Gdv^2 = [I^*] \quad (4.2.)$$

and

$$\vec{e}^* = \vec{r}_u \cdot \vec{g}_u^* = \vec{r}_u \cdot \vec{\xi}_u = \vec{r}_1 \cdot \vec{\xi}_1 \vec{E} = \vec{r}_1 \cdot \vec{\xi}_1 E = 0$$

$$\vec{f}^* = \vec{r}_v \cdot \vec{g}_u^* = \vec{r}_v \cdot \vec{\xi}_u = \vec{r}_2 \cdot \vec{\xi}_1 \sqrt{\vec{E} \vec{G}} = \vec{r}_2 \cdot \vec{\xi}_1 \sqrt{EG} = 0 \quad (4.3)$$

$$\vec{f}'^* = \vec{r}_u \cdot \vec{g}_v^* = \vec{r}_u \cdot \vec{\xi}_v = \vec{r}_1 \cdot \vec{\xi}_2 \sqrt{\vec{E} \vec{G}} = \vec{r}_1 \cdot \vec{\xi}_2 \sqrt{EG} = 0$$

$$\vec{g}^* = \vec{r}_v \cdot \vec{g}_v^* = \vec{r}_v \cdot \vec{\xi}_v = \vec{r}_2 \cdot \vec{\xi}_2 \vec{G} = \vec{r}_2 \cdot \vec{\xi}_2 G = -\left(\frac{1}{r}\right)_2 \frac{K}{q} G,$$

$$\vec{dr} \cdot \vec{dg}^* = \vec{dr} \cdot d\vec{\xi} = -\left(\frac{1}{r}\right)_2 \frac{K}{q} Gdv^2 = [II^*] \quad (4.4.)$$

are found.

2. The first and second fundamental form of the congruence \vec{y}^{**} ,

$$\begin{aligned} \vec{G}^{**} &= \vec{g}_u^{**2} = (-\vec{\xi}_u)^2 = \vec{\xi}_2^2 \vec{E} = \vec{\xi}_1^2 \vec{E} = r^2 \vec{E} \\ \vec{g}^{**} &= \vec{g}_u^{**} \cdot \vec{g}_v^{**} = (-\vec{\xi}_u) \cdot (-\vec{\xi}_v) = \vec{\xi}_1 \cdot \vec{\xi}_2 \sqrt{\vec{E}\vec{G}} = \vec{\xi}_1 \cdot \vec{\xi}_2 \sqrt{\vec{E}\vec{G}} = 0 \\ \vec{g}^{**} &= \vec{g}_v^{**2} = (-\vec{\xi}_v)^2 = \vec{\xi}_2^2 \vec{G} = \vec{\xi}_1^2 \vec{G} = \bar{r}^2 \vec{G}, \end{aligned} \tag{4.5}$$

$$d\vec{G}^{**2} = r^2 \vec{E} du^2 + \bar{r}^2 \vec{G} dv^2 = [\vec{I}^{**}] \tag{4.6}$$

and

$$\begin{aligned} \vec{e}^{**} &= \vec{r}_u \cdot \vec{g}_u^{**} = \vec{r}_u \cdot (-\vec{\xi}_u) = -\vec{r}_1 \cdot \vec{\xi}_1 \vec{E} = -\vec{r}_1 \cdot \vec{\xi}_1 \vec{E} = -\left(\frac{1}{\bar{r}}\right) \frac{\vec{K}}{\vec{q}} \vec{E} \\ \vec{f}^{**} &= \vec{r}_v \cdot \vec{g}_u^{**} = \vec{r}_v \cdot (-\vec{\xi}_u) = -\vec{r}_2 \cdot \vec{\xi}_1 \sqrt{\vec{E}\vec{G}} = -\vec{r}_2 \cdot \vec{\xi}_1 \sqrt{\vec{E}\vec{G}} = 0 \\ \vec{f}^{**'} &= \vec{r}_u \cdot \vec{g}_v^{**} = \vec{r}_u \cdot (-\vec{\xi}_v) = -\vec{r}_1 \cdot \vec{\xi}_2 \sqrt{\vec{E}\vec{G}} = -\vec{r}_1 \cdot \vec{\xi}_2 \sqrt{\vec{E}\vec{G}} = 0 \\ \vec{g}^{**} &= \vec{r}_v \cdot \vec{g}_v^{**} = \vec{r}_v \cdot (-\vec{\xi}_v) = -\vec{r}_2 \cdot \vec{\xi}_2 \vec{G} = -\vec{r}_2 \cdot \vec{\xi}_2 \vec{G} = 0, \end{aligned} \tag{4.7}$$

$$\vec{dr} \cdot d\vec{g}^{**} = -\vec{dr} \cdot d\vec{\xi} = \left(\frac{1}{\bar{r}}\right) \frac{\vec{K}}{\vec{q}} \vec{E} du^2 = ([\vec{II}^{**}]) \tag{4.8}$$

are written.

Also from (4.3) and (4.7), we see that the congruences \vec{y}^* and \vec{y}^{**} are normal congruences.

Since (4.2) and (4.6) are same, we may state the following theorem:

4.2. Theorem

The first fundamental forms or the linear elements of their spherical representations of the congruences \vec{y}^* and \vec{y}^{**} are equal.

From (4.1) which is identical of (4.5), for $\overline{\mathcal{H}}^*$ and $\overline{\mathcal{H}}^{**}$, we find,

$$\overline{\mathcal{H}}^{*2} = \overline{\mathcal{H}}^{**2} = K^2 EG \quad (4.9)$$

Since EG is always different from zero, we may state the following theorem:

4.3. Theorem

Since the surface \vec{x} cannot be developable, the congruences \vec{y}^* and \vec{y}^{**} cannot be cylindrical congruences.

Since the congruences \vec{y}^* , \vec{y}^{**} are normal congruences, for these we may write $(\frac{a}{2} + a^2 = 0, a \neq 0)$ and $(\frac{\bar{a}}{2} + \bar{a}^2 = 0, \bar{a} \neq 0)$ consecutively, which are similar to the conditions $(\bar{q}_1 + q^2 = 0, \bar{q} \neq 0)$ and $(q_2 + q^2 = 0, q \neq 0)$ of the congruences \vec{y} and \vec{y} . Since the limit points of the normal congruences \vec{y}^* and \vec{y}^{**} coincide with their focal points,

1) for \vec{y}^* , from (4.1) and (4.3) we find

$$\bar{l}^*_I = \bar{\rho}^*_I = -\frac{r_2}{Kq} = \frac{1}{r} - \frac{1}{r} \quad (4.10)$$

$$\bar{l}^*_{II} = \bar{\rho}^*_{II} = 0$$

2) for \vec{y}^{**} , from (4.5) and (4.7), we find

$$\bar{l}^{**}_I = \bar{\rho}^{**}_I = -\frac{\bar{r}_1}{K\bar{q}} = \frac{1}{r} - \frac{1}{\bar{r}} = -\left(\frac{1}{\bar{r}} - \frac{1}{r}\right) \quad (4.11)$$

$$\bar{l}^{**}_{II} = \bar{\rho}^{**}_{II} = 0.$$

Their middle points become

1) For \vec{y}^* ,

$$\bar{m}^* = \frac{1}{2} \left(\frac{1}{\bar{r}} - \frac{1}{r} \right), \quad (4.12)$$

2) For \vec{y}^{**} ,

$$\vec{m}^{**} = \frac{1}{2} \left(\frac{1}{r} - \frac{1}{\bar{r}} \right) = -\vec{m}^*. \quad (4.13)$$

If we take

$$\bar{\rho}^*_{\text{I}} - \rho^*_{\text{II}} = \frac{1}{\bar{r}} - \frac{1}{r} = \bar{\rho}^*$$

and

$$\rho^{**}_{\text{II}} - \bar{\rho}^*_{\text{II}} = \frac{1}{\bar{r}} - \frac{1}{r} = \rho^{**} = \bar{\rho}^*,$$

1) Taking the middle surface

$$\vec{m}^* = \vec{r} + \frac{\bar{\rho}^*}{2} \vec{g}^* = \vec{r} + \frac{\bar{\rho}^*}{2} \vec{\xi} \quad (4.14)$$

of \vec{y}^* as the reference surface, the first focal surface generated by the focal point $\bar{\rho}^*$ becomes

$$\vec{k}^* = \vec{m}^* + \frac{\bar{\rho}^*}{2} \vec{g}^* = \vec{x} + \frac{1}{\bar{r}} \vec{\xi} = \vec{r}, \quad (4.15)$$

the second focal surface generated by the focal point ρ^*_{II} becomes

$$\vec{p}^* = \vec{m}^* - \frac{\bar{\rho}^*}{2} \vec{g}^* = \vec{r} = \vec{x} + \frac{1}{r} \vec{\xi} = \vec{z} + \frac{1}{b} \vec{n}. \quad (4.16)$$

2) Taking the middle surface

$$\vec{m}^{**} = \vec{r} + \frac{\bar{\rho}^{**}}{2} \vec{g}^{**} = \vec{r} - \frac{\bar{\rho}^{**}}{2} \vec{\xi} \quad (4.17)$$

of \vec{y}^{**} as the reference surface, the focal surfaces generated by the focal points $\bar{\rho}^{**}_{\text{I}}$ and $\bar{\rho}^{***}_{\text{II}}$ become consecutively,

$$\vec{k}^{**} = \vec{m}^{**} + \frac{\bar{\rho}^{**}}{2} \vec{g}^{**} = \vec{x} + \frac{1}{r} \vec{\xi} = \vec{r} \quad (4.18)$$

$$\vec{p}^{**} = \vec{m}^{**} - \frac{\vec{\rho}^{**}}{2} \quad \vec{g}^{**} = \vec{r} = \vec{x} + \frac{1}{\vec{r}} \vec{z} = \vec{z} + \frac{1}{\vec{\beta}} \vec{n}. \quad (4.19)$$

From these the following theorem may be stated:

4.4. Theorem

1°) The distance between the focal points of the congruence \vec{y}^* and \vec{y}^{**} is equal to the distance between the central points of the base surface \vec{x} , i.e. the focal surfaces of \vec{y}^* and \vec{y}^{**} coincide with the central surface of \vec{x} ($\vec{p}^* = \vec{k}^{**} = \vec{r}$, $\vec{k}^* = \vec{p}^{**} = \vec{r}$).

2°) One focal surface of \vec{y}^* and \vec{y}^{**} each, coincides with the focal surface of \vec{y} and \vec{y} each and become the reference surface ($\vec{p} = \vec{p}^* = \vec{k}^{**} = \vec{r}$, $\vec{p} = \vec{k}^* = \vec{p}^{**} = \vec{r}$).

On the other hand, from the definition of b and (2.3) we find

$$\frac{1}{\vec{b}} - \frac{1}{\vec{b}} = \rho \quad (4.20)$$

and from the definition of $\vec{\beta}$ and (2.7) we find

$$\frac{1}{\vec{\beta}} - \frac{1}{\vec{\beta}} = \bar{\rho}. \quad (4.21)$$

From these, we find

$$\vec{k} = \vec{z} + \frac{1}{\vec{b}} \vec{n}$$

and

$$\vec{k} = \vec{z} + \frac{1}{\vec{\beta}} \vec{n}. \quad (4.23)$$

From 4.4. Theorem 2°, (4.16) and (4.18) we get

$$\vec{p} = \vec{z} + \frac{1}{\beta} \vec{n}$$

$$\vec{\hat{p}} = \vec{z} + \frac{1}{\beta} \vec{\hat{n}}$$

Therefore the following theorem may be stated:

4.5. Theorem

The distance between the focal points of the congruences \vec{y} and $\vec{\hat{y}}$ is equal to the distance between the central points of the surfaces \vec{z} and $\vec{\hat{z}}$ i.e. the focal surfaces of \vec{y} coincide with the central surfaces of \vec{z} , the focal surfaces of $\vec{\hat{y}}$ coincide with the central surfaces of $\vec{\hat{z}}$.

Since the congruences $\vec{\hat{y}}^*$ and $\vec{\hat{y}}^{**}$ are the normal congruences, their principal surfaces which are developable,

1) for $\vec{\hat{y}}^*$

$$EG \frac{r_2 K}{q} dudv = 0, \tag{4.24}$$

2) for $\vec{\hat{y}}^{**}$

$$-EG \frac{\hat{r}_1 K}{\hat{q}} dudv = 0 \tag{4.25}$$

are found. Since $EG \neq 0$, $r_2 \neq 0$, $r_1 \neq 0$ and $K \neq 0$ in (4.24) and (4.25), the following theorem may be stated:

4.6. Theorem

The principal surfaces which are developable of the normal congruences $\vec{\hat{y}}^*$ and $\vec{\hat{y}}^{**}$ are parametric surfaces.

The mean ruled surfaces of the congruences $\vec{\hat{y}}^*$ and $\vec{\hat{y}}^{**}$, we find

1) for $\vec{\hat{y}}^*$

$$\bar{\varepsilon}^* du^2 - \bar{g}^* dv^2 = 0 \quad (4.26)$$

and

2) for \vec{y}^{**}

$$\bar{\varepsilon}^{**} dv^2 - \bar{g}^{**} du^2 = 0. \quad (4.27)$$

Since $\bar{\varepsilon}^* = \bar{\varepsilon}^{**}$ and $\bar{g}^* = \bar{g}^{**}$, the following theorem may be stated:

4.7. Theorem

The mean ruled surfaces of the normal congruences \vec{y}^* and \vec{y}^{**} coincide. As the condition of the congruences \vec{y}^* and \vec{y}^{**} to become isotropic congruences we find

1) for \vec{y}^*

$$-\left(\frac{1}{r}\right)_2 \frac{K}{q} G = 0, \quad (4.28)$$

2) for \vec{y}^{**}

$$-\left(\frac{1}{r}\right)_1 \frac{K}{q} E = 0. \quad (4.29)$$

Since $E \neq 0$, $G \neq 0$, $r_2 \neq 0$, $r_1 \neq 0$, we may state the following theorem:

4.8. Theorem

Since the base surface \vec{x} cannot be developable and canal surface, the congruences \vec{y}^* and \vec{y}^{**} cannot become isotropic congruences.

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