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THE UPPER BOUNDS FOR THE ZEROS OF A POLYNOMIAL WITH REAL OR COMPLEX COEFFICIENTS

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ABSTRACT

In this study, considering matrix norms and companion matrix, we obtained some upper bounds for the roots of a polynomial with real or complex coefficients.

1. INTRODUCTION

Firstly let us give the following definitions [2]:

Definition 1.1. We call a function $\| \cdot \| : M_n \to \mathbf{R}$ a matrix norm if for all A, B $\in M_n$ it satisfies the following axioms:

1) $\|A\| \geq 0$
2) $\|\alpha A\| = |\alpha| \, \|A\|$ for all complex scalars α
3) $\|A + B\| \leq \|A\| + \|B\|$
4) $\|AB\| \leq \|A\| \, \|B\|$

where M_n are $n \times n$ complex matrices.

Definition 1.2. The maximum column sum matrix norm $\| \cdot \|_1$ is defined on M_n by

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

Definition 1.3. The maximum row sum matrix norm $\| \cdot \|_\infty$ is defined on M_n by

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

Definition 1.4. The Euclidean norm $\| \cdot \|_E$ is defined for $A \in M_n$ by

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

Definition 1.5. The spectral norm $\|\cdot\|_2$ is defined for $A \in M_n$ by

$$\|A\|_2 = \max \{\sqrt{\lambda} : \lambda \text{ is an eigenvalue } A^*A\}$$

where A^* is the transpose of the conjugate of A.

Definition 1.6. The spectral radius ρ (A) of a matrix $A \in M_n$ is defined by

$$\rho(A) = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}.$$

Any polynomial f(z) of degree at least 1 can be written in the form $f(z) = \alpha z^k p(z)$, where α is a nonzero constant,

$$p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \tag{1.1}$$

and $a_0 \neq 0$. The roots of $p(z) = 0$ are the nonzero roots of $f(z) = 0$, and they are the roots for which we can give various bounds. In [1] using operator norms, it was given the bounds for the roots of Algebraic Equations.

2. THE STATEMENT OF MAIN RESULTS

Lemma 2.1. The characteristic polynomial of the companion matrix

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

is exactly p(z) and the eigenvalues of C(p) are the same as the roots of $p(z) = 0$.

Proof: If we compute det $[zI - C(p)]$, using cofactors of the first column and use induction then the proof is immediately seen.

Lemma 2.2. If z is any root of $p(z) = 0$ and if $\|\cdot\|$ is any matrix norm on M_n then

$$|z| \leq \|C(p)\|$$

Proof: For any vector $0 \neq x \in C^n$ we write

$$C(p)\, x = zx$$

On the other hand since

$$\|zx\| = |z| \; \|x\| \tag{2.1}$$

and

$$\|C(p)\, x\| \leq \|C(p)\| \; \|x\| \tag{2.2}$$

using (2.1) and (2.2) we get

$$|z| < \|C(p)\|.$$

Theorem 2.1. If z is any root of $p(z) = 0$, then the following statement are satisfied:

1) $|z| \leq 1 + \max \{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$

2) $|z| \leq 1 + \sum_{i=0}^{n-1} |a_i|$

3) $|z| \leq (n-1) + \sum_{i=0}^{n-1} |a_i|$

4) $|z| \leq [(n-1) + \sum_{i=1}^{n-1} |a_i|^2]^{1/2}$

5) $|z| \leq n \max \{1, |a_0|, |a_1|, \ldots, |a_{n-1}|\}$.

Proof: In Lemma 2.1., for the companion matrix, considering the matrix norms $\|\cdot\|_1$, $\|\cdot\|_\infty$, $\|\cdot\| = \sum_{i,j=1}^{n} |a_{ij}|$, $\|\cdot\|_E$, $n\|\cdot\|_\infty$, respectively, we have

1) $|z| \leq \max \{|a_0|, 1 + |a_1|, \ldots, 1 + |a_{n-1}|\}$
$\leq 1 + \max \{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$

2) $|z| \leq \max \{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\}$
$\leq 1 + |a_0| + |a_1| + \ldots + |a_{n-1}|$

3) $|z| \leq (n-1) + |a_0| + |a_1| + \ldots + |a_{n-1}|$

4) $|z| \leq [(n-1) + |a_0|^2 + |a_1|^2 + \ldots + |a_{n-1}|^2]^{1/2}$

5) $|z| \leq n \max \{1, |a_0|, |a_1|, \ldots, |a_{n-1}|\}$

Note 2.1. The bound 2) is obviously poorer than the bound 1). Furthermore the bound 3) is poorer than the bound 2).

Theorem 2.2. If z is a root of $p(z) = 0$, then

$$|z| < [1 + |a_0|^2 + |a_1|^2 + \ldots + |a_{n-1}|^2]^{1/2}$$

Proof. We can write the companion matrix as $C(p) = S + R$, where

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

and

$$R = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

It is easily seen that $S^*R = R^*S = 0$ and $\|S^*S\|_2 = 1$ and $\|R^*R\|_2 = |a_0|^2 + |a_1|^2 + \dots + |a_{n-1}|^2$. Thus we have

$$\begin{aligned} \|C(p)\|_2^2 &= \|C(p)^* \ C(p)\|_2 \\ &= \|(S+R)^* \ (S+R)\|_2 \\ &= \|S^* \ S + R^* \ R\|_2 \\ &< \|S^* \ S\|_2 + \|R^* \ R\|_2 \ . \end{aligned}$$

So the proof follows.

Lemma 2.3. If $q(z) = (z-1)\,p(z)$ and z is a root of $p(z) = 0$, then $|z| < |a_0| + |a_0 - a_1| + \dots + |a_{n-2} - a_{n-1}| + |a_{n-1} - 1|$.

Proof. If $q(z) = (z-1)\,p(z)$, then we have

$$q(z) = z^{n+1} + (a_{n-1}-1)z^n + (a_{n-2}-a_{n-1})z^{n-1} + \dots + (a_0 - a_1)z + a_0.$$

On the other hand considering the matrix norm $\| \cdot \|_\infty$ we get

$$|z| \leq \max\{1, \ |a_0| + |a_0 - a_1| + \dots + |a_{n-2} - a_{n-1}| + |a_{n-1} - 1|\}.$$

In this expression since the second term is not less than 1 we write

$$|z| \leq |a_0| + |a_0 - a_1| + \dots + |a_{n-2} - a_{n-1}| + |a_{n-1} - 1|.$$

Example 2.1. Let us consider

$$f(z) = \frac{1}{n!} z^n + \frac{1}{(n-1)!} z^{n-1} + \dots + \frac{1}{2} z^2 + z + 1$$

which is the n-th partial sum of the power series for the exponential function e^z. Indeed all roots z of $f(z) = 0$ satisfy the inequality $|z| \leq 1 + n!$.

Theorem 2.3. Let the polynomial p(z) be (1.1). If z is a root of p(z) = 0 then

$$|z|\leq \max\{|a_0|\frac{d_n}{d_1}, |a_1|\frac{d_{n-1}}{d_1}+\frac{d_{n-1}}{d_n}, \ldots, |a_{n-2}|\frac{d_2}{d_1}+\frac{d_2}{d_3},$$

$$|a_{n-1}|+\frac{d_1}{d_2}\} \quad (2.3)$$

where for all $d_i > 0$, $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

Proof. Since $\rho(A) = \rho(D^{-1}AD)$ for any nonsingular matrix D, if we consider the norm $\|\cdot\|_1$ of the matrix $D^{-1}C(p)D$ then the proof follows.

Carollary 2.1. If all the coefficients a_k in (1.1) are nonzero and z is a root of p(z) = 0, then

$$|z|\leq \max\{|a_0|, 2|\frac{a_1}{a_2}|, 2|\frac{a_2}{a_3}|, \ldots, 2|\frac{a_{n-1}}{a_n}|\}.$$

Proof. In Theorem 2.3., if we choose $dm \equiv \frac{d_1}{|a_{n-m+1}|}$ where $m = 2, 3, \ldots, n$, instead of d_i and use the inequality (2.3), then the proof is immediately seen.

Corollary 2.2. If z is a root of p(z) = 0, then for any r > 0

$$|z| < \frac{1}{r} + \max_{0\leq k\leq n-1}\{|a_k|\, r^{n-k-1}\}.$$

Proof. In Theorem 2.3., if we choose $d_k = r^k$, $k \equiv 1, 2, \ldots, n$ for some $r > 0$ and use (2.3) then we get

$$|z|\leq \max\{|a_0|\, r^{n-1}, |a_1|r^{n-2}+r^{-1}, |a_2|r^{n-3}+r^{-1}, \ldots, |a_{n-1}|\}$$

$$\leq \frac{1}{r} + \max_{0\leq k\leq n-1}\{|a_k|\, r^{n-k-1}\}.$$

REFERENCES

[1] FUJII M. AND KUBO F., Operator norms as bounds for roots of algebraic equations, Proc. Japon Acad. 49: 805–808, 1973.

[1] TASCI, D., *lp* Matrix Norms, Doctoral Thesis, Konya, 1992.