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SURFACES OF PARALLEL MEAN CURVATURE IN FOUR DIMENSIONAL MANIFOLDS OF CONSTANT CURVATURE

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ABSTRACT

In this paper ve classify the condimension two immersions in constantly curved 4 - manifolds with parallel mean curvature by reproving the Proposition 1 in [1] in the case of $K \equiv 0$.

PRELIMINARIES

Let $i: M^2 \to \bar{M}^4$ (c) be an isometric immerison of a 2–dimensional Riemannian manifold M^2 in a 4– dimensional Riemannian manifold M^4 (c) of constant sectional curvature c and let X and Y be two tangent vector fields on M^2; i.e., two members of Γ (TM^2), the space of smooth sections of TM^2 . If $<, >$ denotes the metric tensor on $T\bar{M}^4$ than that of TM^2 is given by

$$<i_*X, i_*Y> = g(X,Y) \tag{1.1}$$

For all local formulas and computations we may consider i as an imbedding thus identify M^2 with $i(M^2)$ and TM^2 with $i_* (TM^2) \subset T\bar{M}^4$, deleting reference to i and its induced maps wherever possible. As a result, for $X, Y \in T(M^2)_p$ we write $< X, Y >$ for $g(X,Y)$, which we can do via the identification. We consider $T\bar{M}^4$ restricted to the base space M^2 . Let $[\ ]^T$ denote piojection in $T\bar{M}^4$ onto TM^2. Then the normal bundle NM^2 is the bundle whose fibre at p is

$$N(M^2)p = \{x \in T(\bar{M}^4)_p \mid [\times]^T = 0\} \tag{1.2}$$

which is the orthogonal complement (with respect to $<,>$) of $T(M^2)_p$ in $T(\bar{M}^4)_p$. NM^2 has an induced metric, the restriction of $<, >$ to NM^2. We let $[\ ]^N$ denote projection onto NM^2. Let ∇ and $\bar{\nabla}$ be the Riemannian connections of M^2 and $\bar{M}^4$ respectively. ∇ related to $\bar{\nabla}$ by

$$[\bar{\nabla}_X Y]^T = \nabla_X Y \tag{1.3}$$

Then the second fundamental form of the immersion is given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \tag{1.4}$$

and is a section of Γ ($TM^2 \otimes TM^2$, NM^2), the tensor bundle over M^2 whose fibre at p is the space of bilinear maps from $T(M^2)_p \times T(M^2)_p$ to $N(M^2)_p$.

$$B(X,Y) = [\overline{\nabla}_X Y]^N \tag{1.5}$$

is a normal vector field on M^2 and is symmetric on X and Y. Let $N \in \Gamma(TM^2)$; we write

$$\overline{\nabla}_X N = A(N,X) + D_X N \tag{1.6}$$

where $A(N,X)$ and $D_X N$ denote the tangential and normal components of $\overline{\nabla}_X N$. A is a section of Γ ($NM^2 \otimes TM^2$, TM^2) defined by

$$<A(N,X), Y> = -< B(X,Y), N> \tag{1.7}$$

and D is the Riemannian connection on NM^2 induced by the immersion defined by

$$D_X N = [\overline{\nabla} N]^N \tag{1.8}$$

D is easily seen to be compatible with the metric of NM^2. A normal vector field N on M^2 is said to be parallel in the normal bundle if $D_X N = 0$ for all tangent vector fields X. The mean curvature vector H is the section of NM^2 defined by

$$H = \tfrac{1}{2} \text{ trace } B \tag{1.9}$$

The surface M^2 in $\bar{M}^4(c)$ is said to be minimal if $H = 0$ identically. If the mean curvature vector H and the second fundamental form B satisfy

$$< B(X,Y), H > = \lambda < X,Y >$$

for all tangent vector fields X, Y at p with the same λ then M^2 is said to be pseudo-umbilical at p. If M^2 is pseudo - umbilical at every point of M^2, then M^2 is called a pseudo - umbilical surface of $\bar{M}^4$ (c). The curvatures associated with ∇, $\overline{\nabla}$ and D are denoted R, $\bar{R}$ and $\tilde{R}$ respectively. For example, $\tilde{R}$ is given by

$$\tilde{R}(X,Y) N = D_X D_Y N - D_Y D_X N - D_{[X,Y]} N \tag{1.11}$$

$\tilde{R}$, like $\bar{R}$ and R is skew - symmetric on each of NM^2, bilinear in X and Y. Also, as is obvious, $\tilde{R}(X, Y)_p$ depends only on X_p and Y_p. A local orthonormal frame of $T\bar{M}^4$ (resp., TM^2) we mean four (resp., two) sections e_i of $T\bar{M}^4$ (resp., TM^2) defined on an open set $\bar{U}$ (resp., U) such that $<e_i, e_j> = \delta_{ij}$. For an immersed manifold $M^2 \to \bar{M}^4$, we often consider the frame as defined on $\bar{U}\,|\,M^2$. It will be convenient to choose frame of $T\bar{M}^4$ that have the property that $\{e_1, e_2\}$ are sections of $TM^2 \subset T\bar{M}^4$, and $\{e_3, e_4\}$ are sections of NM^2. Such a frame is called an adapted orthonormal frame. Given a basis of coordinate vectors $\left\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right\}$ of TM^2, a completion to a basis of $T\bar{M}^4$ is a choice of two orthonormal sections of $\{e_\alpha\}_{\alpha=3}^4$ of NM^2 $\{e_\alpha\}_{\alpha=3}^4$ is a frame of the normal bundle NM^2. We will call $\left\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, e_3, e_4 \right\}$ an adapted coordinate frame of $T\bar{M}^4$.

For a unit normal section e_α of NM^2 and a frame $\{e_i\}_{i=1}^2$ of TM^2

$$\lambda^\alpha{}_{ij} = < B(e_i, e_j), e_\alpha > \tag{1.12}$$

is the second fundamental form matrix, in the e_α direction, expressed in terms of frame $\{e_i\}_{i=1}^2$ of TM^2. Similarly for a coordinate basis $\left\{ \frac{\partial}{\partial u_i} \right\}_{i=1}^2$

$$L^\alpha{}_{ij} = < B\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right), e_\alpha > \tag{1.13}$$

A normal (or adapted) frame for an immersion $M^2 \to \bar{M}^4$ (c) is said to be on Otsuki frame if $e_3 = H / \|H\|$ where H is the mean curvature vector of the immersion.

SURFACES WITH PARALLEL MEAN CURVATURE VECTOR FIELD

The immersion $M^2 \to \bar{M}^4(c)$ has parallel mean curvature vector field H if H is parallel in the normal bundle. Sometimes this condition will be stated by saying merely that H is parallel. The following lemma is based on the fact that in the case of codimension two, the existence of one parallel vector field implies the existence of another. (Hoffman [2]).

Lemma 1. Let $M^2 \to \bar{M}^4(c)$ be an isometric immersion. If $H \neq O$ parallel then the curvature of the normal connection is zero.

Proof: Since $H \neq O$ parallel $\|H\|$ is a nenzero constant. Let $\{e_\alpha\}_{\alpha=3}^4$ be the Otsuki frame of the normal bundle, that is $e_3 = H/\|H\|$, $<e_3, e_3> = <e_4, e_4> = 1$, $<e_3, e_4> = 0$. If $X \in \Gamma(TM^2)$ then

$$
\begin{aligned}
0 &= X <e_3, e_4> \\
&= <D_X e_3, e_4> + <e_3, D_X e_4> \\
&= <e_3, D_X e_4>, \text{ since } e_3 \text{ is parallel.}
\end{aligned}
$$

But, $2 < D_X e_4, e_4> = 0$ which implies $D_X e_4 = 0$, i.e., e_4 is parallel.

Let $Y \in \Gamma(TM^2)$. For $\alpha = 3,4$

$$
\begin{aligned}
\tilde{R}(X, Y)e_\alpha &= D_X D_Y e_\alpha - D_Y D_X e_\alpha - D_{[X,Y]} e_\alpha \\
&= 0, \text{ since } e_\alpha \text{ is parallel.}
\end{aligned}
$$

Hence, $\tilde{R} = 0$ identically.

As a corollary of this we have

Corollary 1. Let $M^2 \to \bar{M}^4(c)$ be an isometric immersion. If $H \neq O$ parallel then the second fundamental forms are simultaneously diagonalizable.

Proof: The proof follows from the following fact (Hoffman [2])

$$\tilde{R} = 0 \text{ at } p \Leftrightarrow A_3 A_4 = A_4 A_3 \text{ at } p$$

which is precisely the condition for simultaneous diagonalization, where $A_\alpha = A(e_\alpha, -)$, $\alpha = 3,4$. $\tilde{R} = 0$ identically by Lemma 1.

Proposition 1. Let $M^2 \to \bar{M}^4(c)$ be an isometric immersion given locally in conformal coordinates (u_1, u_2) with conformal parameter E, i.e, $< \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} > = E\,\delta_{ij}$ and $ds^2 = E\,(du_1{}^2 + du_2{}^2)$.

If $H \neq O$ parallel and $\{e_\alpha\}_{\alpha=3}^4$ is the Otsuki frame of the normal bundle then

$$\varphi_3 = \frac{L^3{}_{11} - L^3{}_{22}}{2} - i\, L^3{}_{12} \tag{2.1}$$

and

$$\varphi_4 = \frac{L^4_{11} - L^4_{22}}{2} - i\, L^4_{12} \qquad (2.2)$$

are real analytic functions of $z = u_1 + iu_2$, where $L^\alpha_{ij} = < B \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right), e_\alpha >$, $\alpha = 3, 4$. Furthermore, either $\varphi_3 \equiv 0$ or $\varphi_4 = \mathcal{K}\, \varphi_3$ where $\mathcal{K}$ is a real constant.

Proof: Hoffman [2].

In Theorem 1 and Theorem 2, $\bar{M}^4$ (c) will be assumed to be simply connected an complete.

Theorem 1. Let $M^2 \to \bar{M}^4(c)$, $c > 0$, be an isometric immersion. If $H \neq O$ parallel and $K \equiv 0$ then M^2 lies in an affine 3- plane π^3 in E^4 (in the case $c = 0$) or in a great or small 3-sphere (in the case $c > 0$)

Proof: First assume $c = 0$ and therefore $M^2 \to E^4$. The proof will follow from the following lemma: (Hoffman [2]).

Lemma 2. Suppose $M^2 \to E^{2+k}$ has an r- dimensional distribution $\mathcal{P}$ in NM^2 such that

(a) The range of B is in $\mathcal{P}$

(b) If V is a smooth section of $\mathcal{P}$ then $D_X V \in \mathcal{P}$ for all $V \in \Gamma(TM^2)$

Then M^2 lies in a $(2 + r)$ — plane π^{2+r} in E^{2+k}.

Now we return to the proof of the Theorem 1. Let $\{e_\alpha\}^4_{\alpha=3}$ be the Otsuki frame of the normal bundle. Since $K \equiv 0$, M^2 is isometric to the plane and we may choose conformal coordinates locally on M^2 with $E = 1$. Hence $(\lambda^\alpha_{ij}) = (L^\alpha_{ij})$ for $\alpha = 3,4$. Proposition 1 and Corollary 1 give (λ^3_{ij}) and (λ^4_{ij}) as constant diagonalized matrices. Namely, if $\varphi_3 \not\equiv 0$ then by Proposition 1, $\varphi_4 = \mathcal{K}\, \varphi_3$ for a real constant $\mathcal{K}$. From the equation,

$$E^2 = (\|H\|^2 - (K—c)) = |\varphi_3|^2 + |\varphi_4|^2$$

((3.1) in [2]) we obtain that $|\varphi_3|$ is constant. Therefore φ_3 is constant and after a possible rotation of coordinates it may be assumed to be real. If $\varphi_3 = \gamma$ then $\lambda^3_{11} = \|H\| + \gamma$ and $\lambda^3_{22} = \|H\| - \gamma$ since $\lambda^3_{11} + \lambda^3_{22} = 2\, \|H\|$ and since $\lambda^3_{12} = \lambda^3_{21} = 0$ by Corollary 1. Similarly; $\lambda^4_{11} = \mathcal{K}\gamma$ and $\lambda^4_{22} = - \mathcal{K}\gamma$ since $\lambda^4_{11} + \lambda^4_{22} = 0$ and $\lambda^4_{12} = \lambda^4_{21} = 0$ by Corollary 1. All the real constants γ, $\mathcal{K}$ and $\|H\|$ are related to each other as $\gamma^2 = \|H\|^2/1 + \mathcal{K}^2$.

If $\mathcal{K} = 0$ then define,

$$\mathcal{D} = \{e \in NM^2 \mid < e, e_4 > = 0\} \tag{2.3}$$

$\mathcal{D}$ is a 1 - dimensional distribution in NM^2. Since, $0 = \lambda^4{}_{ij} = < B(e_i,e_j), e_4 >$, the range of B is in $\mathcal{D}$. If V is a smooth section of $\mathcal{D}$; for all p in M^2, V_p is in $\mathcal{D}_p$. Hence $< V_p, e_4 > = 0$ with

$$0 = <D_X V_p, e_4> + <V_p, D_X e_4> = <D_X V_p, e_4> \tag{2.4}$$

which implies $D_X V$ is in $\mathcal{D}$.

If $\mathcal{K} \neq 0$ then we may put $\mathcal{K} = -\tan\alpha$ then $\tilde{\lambda}_{ij} = \lambda I$ with $\tilde{e} = (\sin\alpha)e_3 + (\cos_a)e_4$ and $\lambda = \sin\alpha \; \|H\|$.

Hence $\bar{\varphi}_3 \equiv 0$.

All that remains to complete the proof of this theorem is to study for the case $c > 0$. We do that by reducing the case to the euclidian case $c = 0$. For $c > 0$ we take as a model for $\bar{M}^4(c)$ the hypersurface

$$S^4(1\sqrt{c}) = \left\{ x \in E^5 \mid \|x\|^2 = \frac{1}{c} \right\} \subset E^5 \tag{2.5}$$

Let $\{e_\alpha\}^4{}_{\alpha=3}$ be the Otsuki frame of the normal bundle. By Proposition 1 and Corollary 1, the second fundamental forms in the e_3 and e_4 directions are constant diagonalized matrices. If we let e_5 be the normal to the sphere in E^5, then the second fundamental form of the composed immersion $M^2 \rightarrow S^4(1|\sqrt{c}) \rightarrow E^5$ in the e_5 direction is also a constant diagonalized matrix. Hence we can find real constants λ, μ, υ such that,

$$\lambda(\lambda^3{}_{ij}) + \mu(\lambda^4{}_{ij}) + \upsilon(^5{}_{ij}) = 0 \; \Lambda \; \lambda^2 + \mu^2 + \upsilon^2 = 1$$

The unit normal vector field $\lambda e_3 + \mu e_4 + \upsilon e_5 = \tilde{e}$ is parallel since e_3, e_4 and e_5 are parallel. Let

$$\mathcal{D} = \{e \in NM^2 \mid \|e\| = 1 \; \Lambda \; <e, \tilde{e}> = 0\} \tag{2.6}$$

$\mathcal{D}$ is a 2 - dimensional distribution which contains the image of second fundamental form of the immersion $M^2 \rightarrow E^5$. It satisfies condition (b) of Lemma 2 since $\tilde{e}$ is parallel. Therefore Lemma 2 applies $M^2 \rightarrow \pi^4$ where π^4 is some affine plane. If π^4 passes through the orijin, $M^2 \rightarrow S^4(1\sqrt{c}) \subset \pi^4$ lies in a great 3 - sphere and in a small 3 - sphere other wise.

Remark: Here, we are reducing the codimension of an immersion $M^2 \rightarrow \bar{M}^4(c)$ which is not pseudo-umbilical. It is known that pseudo-umbilical immersions with parallel nonzero mean curvature lie minimally in hyperspheres and flat mimimal surfaces in 3-spheres must be pieces of a Clifford torus.

HOFFMAN SURFACES

On an analytic function $\varphi \not\equiv 0$ of $z = u^2 + i\, u_1$, defined in a neighborhood of the origin in the (u_1, u_2) - plane, and constants α, β with $\alpha > 0$ Hoffman proved that, up to euclidian motions and isothermal coordinates $E(u_1, u_2)$, locally there exist ine and only one surface in $\bar{M}^4(c)$, denoted by $M^4(\varphi, \alpha, \beta)$, with parallel mean curvature vector H such that $\alpha = \|H\|$ and $\varphi = \varphi_3$, $\beta\varphi = \varphi_4$ where φ_3 and φ_4 are given in Proposition 1. These surfaces are easy to check that they are contained in either in an affine 3 - space or in a great or small 3 - sphere of $\bar{M}^4(c)$ and they are neither minimal surfaces in $\bar{M}^4(c)$ nor minimal surfaces of hyperspheres of $\bar{M}^4(c)$.

Theorem 2. (CLASSIFICATION), Let $M^2 \to \bar{M}^4(c)$, $c > 0$, be an isometric immersion with parallel mean curvature vector field H. Then M^2 is one of the following surfaces:

(a) Minimal surfaces of $\bar{M}^4(c)$,

(b) Minimal surfaces of hyperspheres of $\bar{M}^4$ (c),

(c) Surfaces in an affine 3 - space or in a great or small 3 - sphere of $\bar{M}^4(c)$ and locally given by Hoffman surfaces.

Proof. Since, H is parallel, $\|H\|$ is constant. If $\|H\| = 0$ then M^2 lies minimally in $\bar{M}^4(c)$. If $\|H\| \neq 0$ and M^2 does not lie minimally in a hypersphere of M^4 (c) then by Proposition 1 in [1] M^2 is contained in an affine 3 - space or in a great or small 3 - sphere of $\bar{M}^4$ (c). If M^2 is contained in an affine3 - space or in great or small 3 - sphere of $\bar{M}^4$ (c) with parallel mean curvature vector H which is neither a minimal surface of $\bar{M}^4$ (c) nor a minimal surface of a hypersphere of $\bar{M}^4$ (c) then by Proposition 1 we see that there exist a triple (Φ, α, β) in which $\Phi \not\equiv 0$ is an analytic quadratic differential given by $\Phi = \varphi_3\, dz^2$, $\alpha = < H, H >^{1/2}$ and β is real constant given by $\varphi_4 = \beta\, \varphi_3$. Let p be a point in M^2 and U be a coordinate neighborhood of P with isothermal coordinates (u_1, u_2). We may construct a Hoffman surface $M(\varphi, \alpha, \beta)$ in $\bar{M}^4$ (c) with φ, α, β satisfying the properties in the introduction of this section. Since locally, this Hoffman surface is unique up to euclidian motions of E^4, the surface M^2 around p must coincide with the Hoffman surface M (φ, α, β).

Remark: If M^2 lies minimally in some hypersphere then its mean curvature vector must be the same as that of hypersphere. Therefore, it must be pseudo - umbilical at each point. But this impies $\varphi_3 = \varphi = 0$.

REFERENCES

1. CHEN, B. YEN, On the surface with parallel mean curvature vector, Indiana University Mathematics Journal, Vol. 22, No. 7 (1973), 655–666.

2. HOFFMAN, D., Surfaces of constant mean curvature in manifolds of constant curvature, J. Differential Geometry, 8 (1973), 161–176.