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ON APPROXIMATION OF ENTIRE HARMONIC FUNCTIONS IN R^3 WITH INDEX - PAIR (p, q)

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ABSTRACT

The authors have defined approximation error for harmonic functions in D_R, $0 < R < \infty$ the class of all harmonic functions H in R^3, that are regular in the open ball D_R of radius R centered at the origin and are continuous on the closure $\bar{D}_R$ of D_R. Necessary and sufficient conditions, in terms of the rate of decay of the approximation error E_n (H, R), such that $H \in D_R$ has analytic continuation as an entire harmonic functions having (p, q) –order ρ and (p, q)–type T, have been obtained.

INTRODUCTION

The harmonic functions in R^3 are the solutions of the Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \frac{\partial^2 H}{\partial x_3^2} = 0. \tag{0.1}$$

A harmonic function H, regular about the origin, can be expanded as

$$H \equiv H(r, \Theta, \varnothing) = \sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a^{(2)}_{nm} \operatorname{Cos} m\varnothing + a^{(2)}_{nm} \operatorname{Sin} m\varnothing) P^m_n(\operatorname{Cos}\Theta) \tag{0.2}$$

where $x_1 = r\operatorname{Cos}\Theta$, $x_2 = r\operatorname{Sin}\Theta\operatorname{Cos}\varnothing$, $x_2 = r\operatorname{Sin}\Theta\operatorname{Sin}\varnothing$ and $P^m_n(t)$ are associated Legendre's functions of first kind of degree m and order n. A harmonic polynomial of degree k is a polynomial of degree k in x_1, x_2 and x_3 which satisfies (0.1).

A harmonic function H is said to be regular in $D_R = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 < R^2\}, 0 < R \leq \infty$, if the series (0.2) converges uniformly on compact subset of D_R. A harmonic function H is called entire if it is regular in D_∞, The concepts of the index–pair (p, q), $p \geq q \geq 1$, (p, q)– order and (p, q)–type etc. of an entire function were introduced by Juneja et al. ([4], [5]). Thus if we denote by $\log^{[p]} x$ the quantity log log ... log x, where logarithm is taken p times, then an entire harmonic function H is said to be (p, q)–order ρ if it is of index–pair (p, q) and

$$\limsup_{r\to\infty} \frac{\log^{[p]} M(r, H)}{\log^{[q]} r} = \rho(p, q) \equiv \rho(H), b \leq \rho \leq \infty. \tag{0.3}$$

Here $b = 1$ if $(p, q) = (p, p)$, $p = 2, 3, \ldots$ and $b = 0$ otherwise.

The entire harmonic function H having (p, q)–order ρ, $b < \rho < \infty$, is said to be of (p, q)–type T and lower (p, q)–type t if

$$\lim_{r\to\infty} {\sup \atop \inf} \frac{\log^{[p-1]} M(r, H)}{(\log^{[q-1]} r)^{\rho}} = {T(p, q) \atop t(p, q)} \equiv {T(H) \atop t(H)} \tag{0.4}$$

where $0 \leq t \leq T \leq \infty$, and

$$M(r, H) = \max_{x_1^2 + x_2^2 + x_3^2 = r^2} |H(x_1, x_2, x_3)|.$$

Fryant [2] related ρ and T of an entire harmonic function H with the rate of decrease of coefficients $a_{nm}^{(i)}$ in (0.2), $i = 1, 2$. Analogous results for the solutions of (0.1) which depend only on the variables $x = x_1$ and $y = (x_2^2 + x_3^2)^{1/2}$ have been found in Fryant [1] and Gilbert (3, Theorem 4.3.4).

Let H_R, $0 < R < \infty$, denote the class of all harmonic functions H regular in D_R and continuous on $\bar{D}_R$, the closure of D_R. For $H \in H_R$, let $E_n(H, R)$, the error in approximating the function H by harmonic polynomials of degree at most n in uniform norm, be defined as

$$E_n(H, R) = \inf_{g\in\pi_n} \| H-g \|_R \tag{0.5}$$

where π_n consist of all harmonic polynomials of degree at most n and

$$\| H-g \|_R = \max_{(x_1, x_2, x_3) \in \bar{D}_R} | H(x_1, x_2, x_3) - g(x_1, x_2, x_3) |.$$

Let $H \in H_R$ (class of all harmonic functions H in R^3). Kapoor and Nautiyal [6] have proved the following

Theorem. Let $H \in H_R$. Then H has analytic continuation as an entire harmonic function of order ρ ($0 < \rho < \infty$) and type T ($0 < T < \infty$), if and only if,

$$\limsup_{n \to \infty} n\,(E_n(H, R))^{\rho/n} = e\rho T R^{\rho}.$$

In this paper we have extended above theorem for entire harmonic function of (p, q)–growth. We have also obtained analogous result for lower (p, q)–type of entire harmonic functions. Finally, we have studied the growth of the coefficients of polynomial expansion of entire harmonic function with index–pair (p, q) in terms of approximation error. The following notation is frequently used in the sequel:

Notation:

$$F_{[r]}(x) = \prod_{i=0}^{r} \exp^{[i]} x; \quad \triangle_{[r]}(x) = \prod_{i=0}^{r} \log^{[i]} x$$

$$F_{[-r]}(x) = \frac{x}{\triangle_{[r-1]}(x)}, \quad \triangle_{[-r]}(x) = \frac{x}{F_{[r-1]}(x)}, \quad r = 0, \pm 1,$$

AUXILIARY RESULTS

In this section we give some lemmas that are used in proving Theorems 1 and 2.

Lemma 1.1. Associated Legendre's functions $p_n^m(t)$ satisfy

$$\max_{-1 \le t \le 1} |P_n^m(t)| \le K\,[(n+m)!/(n-m)!]^{1/2}, \tag{1.1}$$

where K is a constant independent of n and m.

Lemma 1.2. Let $H \in H_R$ be entire and $r' > 1$. Then, for all $r > 2r'R$ and all sufficiently large values of n, we have

$$E_n(H, R) \le \bar{K} M(r, H)\,(r' R/r)^{n+1}.$$

Here $\bar{K}$ is a constant.

Lemma 1.3. Let $H \in H_R$. Then, for any $R_* < R$ and $n \ge 1$, we have

$$R_* \max_{m, i}\left[|a_{nm}^{(i)}| \left(\frac{(n+m)!}{(n-m)!}\right)^{1/2} \right] \le K_0 (2n+1) E_{n-1}(H, R),$$

where K_0 is a constant.

The proofs of these results can be found in [6, pp. 1026–27].

Lemma 1.4. Let $H \in H_R$. Then for any $R_* < R$ and $n \ge 1$, there exists an entire function h(z) such that

$$h(z) = \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R) \left(\frac{Z}{R_*}\right)^n, \text{ and}$$

$$M(r, H) \leq |a_{00}^{(1)}| + KK_0 m(r, h), \text{ where } m(r, h) = \max_{|z| \leq r} |h(z)|.$$

Proof. For $H \in H_R$, using (0.2), Lemma 1.2 and Lemma 1.3, we have

$$|H| \leq |\sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} (a_{nm}^{(1)} \text{Cos} m\varnothing + a_{nm}^{(1)} \text{Sin} m\varnothing) P_n^m (\text{Cos}\Theta)|,$$

or

$$M(r, H) \leq |a_{00}^{(1)}| + K \sum_{n=}^{\infty} (2n+1) r^n \max_{m, i} \left[|a_{nm}^{(i)}| \left(\frac{(n+m)!}{(n-m)!}\right)^{1/2} \right].$$

$$\geq |a_{00}^{(1)}| + KK_0 \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R) \left(\frac{r}{R_*}\right)^n,$$

for some $R_* < R$. Hence

$$M(r, H) \leq |a_{00}^{(1)}| + KK_0 m(r, h)$$

where

$$h(z) = \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}(H, R) \left(\frac{Z}{R_*}\right)^n.$$

Since $\lim_{n \to \infty} (E_n(H, R))^{1/n} = 0$, $h(z)$ is an entire function of a single complex variable z and $m(r, h) = \max_{|z| \leq r} |h(z)|$.

2. MAIN RESULTS

Theorem 2.1. Let $H \in H_R$. Then H has analytic continuation as an entire harmonic function of (p, q)–order ρ ($b < \rho < \infty$) and (p, q)–type T ($0 \leq T \leq \infty$), such that

$$\limsup_{n \to \infty} \frac{\log^{[p-2]} n}{[\log^{[q-1]} (E_n(H, R))^{-1/n}]^{\rho - A}} = \frac{T}{M}. \tag{2.1}$$

where

$$M = \begin{cases} \dfrac{1}{\rho e R^{\rho}} & \text{for } (p, q) = (2, 1), \\ \dfrac{(\rho-1)^{\rho-1}}{\rho^{\rho}} & \text{for } (p, q) = (2, 2), \\ 1. & \text{otherwise.} \end{cases} \tag{2.2}$$

and

$$A = \begin{cases} 1 & \text{for } (p, q) = (2, 2), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let $K > T$. By the definition of the (p, q) type of H there exists an $R_2 = R_2(K) > R_1$ such that

$$\frac{\log^{[p-1]} M(r, H)}{(\log^{[q-1]} r)^{\rho}} \leq K \quad \text{for} \quad r \geq R_2,$$

or

$$M(r, H) \leq \exp^{[p-1]} [K (\log^{[q-1]} r)^{\rho}]. \tag{2.4}$$

By Lemma 1.3, we have

$$E_n(H, R) \leq \bar{K} M(r, H) \left(\frac{r' R}{r}\right)^{n+1} \tag{2.5}$$

For (p, q) = (2, 1) we proceed on the lines of Kapoor and Nautiyal [6] to get

$$n\,(E_n(H, R))^{\rho/n} \geq e\rho K R^{\rho}. \tag{2.6}$$

Now for (p, q) = (2, 2), from (2.4) we have

$M(r, H) \leq \exp [K (\log r)^{\rho}]$, and consequently

$$E_n(H, R) \leq \bar{K} \exp [K (\log r)^{\rho}] \left(\frac{r' R}{r}\right)^{n+1}$$

Let $N > n_0$ be so large that $\exp \left(\frac{n}{K\rho}\right)^{1/\rho-1} > R_2$, for $n \geq N$.

Choosing

$r = \exp \left(\frac{n}{K\rho}\right)^{1/\rho-1}$ in above inequality, we get

$$E_n(H, R) \leq \frac{\bar{K} \exp \left[\left(\frac{n}{\rho}\right)^{\rho/\rho-1} \frac{1}{K^{1/\rho-1}}\right] (r' R)^{n+1}}{\left\{\exp \left[\left(\frac{n}{K\rho}\right)^{1/\rho-1}\right]\right\}^{n+1}}$$

$$\Rightarrow \log E_n(H, R) \le \log \bar{K} + \left(\frac{n}{\rho}\right)^{\rho/\rho-1} \frac{1}{K^{1/\rho-1}}$$

$$+ (n+1) \log (r' \; R) - (n+1) \left(\frac{n}{K\rho}\right)^{1/\rho-1}$$

$$\Rightarrow -\frac{1}{n} \log E_n(H, R) \ge \left(\frac{n}{K\rho}\right)^{1/\rho-1} - \left(\frac{1}{\rho}\right)^{\rho/\rho-1} \left(\frac{n}{K}\right)^{1/\rho-1}$$

$$+ \frac{1}{n} \left(\frac{n}{K\rho}\right)^{1/\rho-1} - \log (r' \; R) - \frac{1}{n} \log (K r' \; R)$$

$$= \left(\frac{n}{K\rho}\right)^{1/\rho-1} \left[1 - \frac{1}{\rho} + \frac{1}{n}\right] - \frac{1}{n} \log (K r' \; R) - \log (r' \; R).$$

$$\Rightarrow [\log (E_n(H, R))^{-1/n}]^{\rho-1} \ge \frac{n}{K\rho}\left[\left(\frac{\rho-1}{\rho}\right)(1 + o(1)\right]^{\rho-1}$$

for sufficiently large n,

$$\frac{n}{[\log (E_n(H, R))^{-1/n}]^{\rho-1}} \le K \frac{(\rho)^{\rho}}{(\rho-1)^{\rho-1}} (1 - o(1)).$$

Proceeding to limits we get

$$\limsup_{n\to\infty} \frac{n}{[\log (E_n(H, R))^{-1/n}]^{\rho-1}} \le \frac{K}{M}. \tag{2.7}$$

For $(p, q \ne (2,1)$ and $(2, 2)$, let $N > n_0$ be large such that

$$\exp^{[q-1]} \left[\frac{\log^{[p-2]}(n / K\rho)}{K}\right]^{1/\rho} > R_2 \text{ for } n \ge M.$$

Choosing

$$r = \exp^{[q-1]} \left[\frac{\log^{[\rho-2]}\left(\frac{n}{K\rho}\right)}{K}\right]^{1/p} \text{ in (2.4) and (2.5), we get,}$$

$$E_n(H, R) \le \frac{\bar{K} \exp\left(\frac{n}{K\rho}\right)(r' \; R)^{n+1}}{\left\{\exp^{(q-1)}\left[\frac{\log^{(p-2)}(n / K\rho)}{K}\right]^{1/p}\right\}^{n+1}},$$

or

$$\log E_n (H, R) \leq \log \bar{K} + \frac{n}{K\rho} + (n+1) \log (r' R)$$

$$- (n+1) \exp^{[\rho-2]} \left[\frac{\log^{[p-2]} \left(\frac{n}{K\rho}\right)}{K} \right]^{1/\rho},$$

or

$$(\log E_n (H, R))^{-1/n} \geq \exp^{[q-2]} \left[\frac{\log^{[p-2]} (n / K\rho)}{K} \right]^{1/\rho} [1 - o(1)]$$

for sufficiently large values of n,

$$[\log^{[q-1]} (E_n (H, R))^{-1/n}]^{\rho} \geq \left[\frac{\log^{[p-2]} (n / K\rho)}{K} \right] [1 - o(1)]^{\rho},$$

or, since $p > 2$,

$$K \geq \frac{\log^{[p-2]} n}{[\log^{[q-1]} (E_n (H, R))^{-1/n}]^{\rho}} [1 - o(1)]^{\rho}.$$

Proceeding to limits we get

$$\limsup_{n \to \infty} \frac{\log^{[p-2]}}{[\log^{[q-1]} (E_n (H, R))^{-2/n}]^{\rho}} \leq K. \qquad (2.8)$$

Since (2.6), (2.7) and (2.8) are valid for every $K > T$, therefore it follows that

$$\limsup_{n \to \infty} \frac{\log^{[p-2]} n}{[\log^{[q-1]} (E_n (H, R))^{-1/n}]^{\rho - A}} \leq \frac{T}{M}. \qquad (2.9)$$

To prove reverse inequality, using Lemma 1.4, we have

$$\limsup_{n \to \infty} \frac{\log^{[p-2]} n}{[\log^{[q-1]} (E_n H, R))^{-1/n}]^{\rho - A}} \geq \frac{T}{M}. \qquad (2.10)$$

(2.9) and (2.10) taking together prove the theorem i.e., the result (2.1).

Theorem 2.2. Let $H \in H_R$. Then, H has analytic continuation as an entire harmonic function of (p, q)-order ρ $(b < \rho \to \infty)$ and lower (p, q)-type t $(0 \leq t \leq \infty)$ such that

$$\liminf_{n\to\infty} \frac{\log^{[p-2]} n}{[\log^{[q-1]} [E_n (H, R))^{-1/n}]\, \rho^{-A}} = \frac{t}{M}, \tag{2.11}$$

where A and M have their usual meaning.

Proof. Let

$$\liminf_{n\to\infty} \frac{\log^{[p-2]} n}{[\log^{[q-1]} (E_n (H, R))^{-2/n}]\, \rho^{-A}} = J.$$

Then for any $\varepsilon > 0$, there exist $n \geq n_0$ such that

$$\frac{\log\, [^{p-2}] n}{[\log^{[q-1]} (E_n (H, R))^{-1/n}]\, \rho^{-A}} \geq J-\varepsilon,$$

or

$$E_n (H, R) \geq \left\{ \exp^{[q-1]} \left[\frac{\log^{[p-2]} n}{J-\varepsilon}\right]^{1/\rho-A} \right\}^{-n} \tag{2.12}$$

For $(p, q) = (2, 1)$,

$$E_n (H, R) \geq \left(\frac{n}{J-\varepsilon}\right)^{-n/\rho},$$

or

$$\bar{K} M (r, H) \left(\frac{r' R}{r}\right)^{n+1} \geq \left(\frac{n}{J-\varepsilon}\right)^{-n/\rho},$$

or

$$\log M (r, H) \geq -n/\rho \log \left(\frac{n}{J-\varepsilon}\right) + (n+1) \log (r/r' R) - \log \bar{K}.$$

Choosing

$$r = \left(\frac{ne (r' R)^{\rho}}{J-\varepsilon}\right)^{1/\rho}; \text{ we get}$$

$$\log M (r, H) \geq - \frac{n}{\rho} \log \left(\frac{n}{J-\varepsilon}\right) + \frac{(n+1)}{\rho} [\log \left(\frac{n}{J-\varepsilon}\right) +$$

$$\log e (r' R)^{\rho}] - (n+1) \log (r' R) - \log \bar{K}.$$

$$= \frac{n}{\rho} + \frac{1}{\rho} \left[\log \left(\frac{ne}{J-\varepsilon}\right)\right] - \log \bar{K}$$

$$= \frac{n}{\rho} [1 + o (1)] - 0(1), \text{ for sufficiently large } n.$$

$$= \frac{(r/r')^{\rho}(J-\varepsilon)}{\rho e R^{\rho}} [1 + o(1)] - 0(1),$$

$$\frac{\log M(r, H)}{r^{\rho}} \geq \frac{(J-\varepsilon)}{\rho e R^{\rho}} \frac{(r/r')^{\rho}}{r^{\rho}} [1 + o(1)].$$

Proceeding to limit, as $r \to \infty$, we get since $r' > 1$ is arbitrary,

$$t \geq M.J. \tag{2.13}$$

Again for (p, q) = (2, 2), from (2.12) we have

$$E_n (H, R) \geq \left\{ \exp \left(\frac{n}{J-\varepsilon}\right)^{1/\rho-1} \right\}^{-n},$$

$$\bar{K}M (r, H) \left(\frac{r' R}{r}\right)^{n+1} \geq \left\{ \exp \left(\frac{n}{J-\varepsilon}\right)^{1/\rho-1} \right\}^{-n}$$

$$\log M (r, H) \geq -(n)^{\rho/\rho-1} \left(\frac{1}{J-\varepsilon}\right)^{1/\rho-1} + (n+1) \log \left(\frac{r}{r' R}\right) - \log \bar{K}.$$

Choosing

$$\frac{r}{r' R} = \exp \{(\rho/\rho-1)(n/J-\varepsilon)^{1/\rho-1}\},$$

we get

$$\log M (r, H) \geq - \frac{(n)^{\rho/\rho-1}}{(J-\varepsilon)} + (n+1) \left[\frac{\rho}{\rho-1} \left(\frac{n}{J-\varepsilon}\right)^{1/\rho-1}\right] - \log \bar{K}.$$

$$= \frac{n^{\rho/\rho-1}}{(J-\varepsilon)^{1/\rho-1}} \frac{1}{\rho-1} [1 + o(1)] - O(1) \text{ as } n \to \infty.$$

$$= (J-\varepsilon) \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}} . (\log (r/r' R))^{\rho} [1+o(1)] - 0(1)$$

$$\frac{\log M(r, H)}{(\log r)^{\rho}} \geq \frac{(\rho-1)^{\rho-1}}{\rho^{\rho}} (J-\varepsilon) \frac{(\log (r/r' R))^{\rho}}{(\log r)^{\rho}} [1 + o(1)] - o(1).$$

Proceeding to limits we get,

$$t \geq M.J. \tag{2.14}$$

Now for (p, q) ≠ (2, 1) and (2.2). From (2.12) and Lemma 2, we have

$$\bar{K}M(r, H)\left(\frac{r'R}{r}\right)^{n+1} \geq \left\{\exp^{[q-1]}\left[\frac{\log^{[p-2]}n}{J-\varepsilon}\right]^{1/\rho}\right\}^{-n}$$

or

$$\log M(r, H) \geq -n \exp^{[q-2]}\left[\frac{\log^{[p-2]}n}{J-\varepsilon}\right]^{1/\rho} + (n+1)\log\left(\frac{r}{r'R}\right)$$

$$- \log \bar{K}.$$

Choosing

$$\frac{r}{r'R} = \exp\left\{1 + \exp^{[q-2]}\left[\frac{\log^{[p-2]}n}{J-\varepsilon}\right]^{1/\rho}\right\}.$$ We get

$$\log M(r, H) \geq n+1+\exp^{[q-2]}\left[\frac{\log^{[p-2]}n}{J-\varepsilon}\right]^{1/\rho} - \log \bar{K}$$

$$= \exp^{[p-2]}\left\{(J-\epsilon)\left(\log^{[q-1]}\left(\frac{r}{er'R}\right)\right)^{\rho}\right\}[1+o(1)]-o(1).$$

$$\frac{\log^{[p-1]} M(r, H)}{(\log^{[q-1]} r)^{\rho}} \geq (J-\varepsilon)\frac{\left(\log^{[q-1]}\left(\frac{r}{er'R}\right)\right)^{\rho}}{(\log^{[q-1]} r)^{\rho}}$$

$$[1 + o(1)] - o(1)$$

Proceeding to limits we get

$$t \geq J. \tag{2.15}$$

Combining the results (2.13), (2.14) and (2.15) we get

$$\frac{t}{M} \geq J. \tag{2.16}$$

To prove reverse inequality, using Lemma 1.4, we get

$$\frac{t}{M} \leq J. \tag{2.17}$$

(2.16), (2.17) taking together prove the theorem i.e., the result (2.11).

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