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SEPERATION PROPERTIES IN CATEGORIES OF PREBORNOLOGICAL SPACES AND BORNOLOGICAL SPACES

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ABSTRACT

In this paper, an explicit characterization of each of the separation properties T_0, T_1, PreT_2, and T_2 is given in the topological categories of Prebornological Spaces and Bornological Spaces. Moreover, specific relationships that arise among the various T_0, PreT_2, and T_2 structures are examined in these categories.

1. INTRODUCTION

Let E be a category and Sets be the category of sets.

1.1. Definition. A Functor $U: E \rightarrow$ Sets is said to be topologîcal or E is a topological category over Sets iff the following conditions hold:

1. U is concrete i.e. faithful (U is mono on hom sets) and amnestic (if $U(f) = id$ and f is an isomorphism, then $f = id$).

2. U has small fibers i.e. $U^{-1}(b)$ is a set for all b in Sets.

3. For every U–source, i.e. family $g_i: b \rightarrow U(X_i)$ of maps in Sets, there exists a family $f_i: X \rightarrow X_i$ in E such that $U(f_i) = g_i$ and if $U(h_i: Y \rightarrow X_i) = kg_i: UY \rightarrow b \rightarrow U(X_i)$, then there exists a lift $\bar{k}: Y \rightarrow X$ of $k: UY \rightarrow UX$ i.e. $U(\bar{k}) = k$. This latter condition means that every U–source has an initial lift. It is well known, see [3] p. 125 or [5] p. 278, that the existence of initial lifts of arbitrary U–source is equivalent to the existence of final lifts (the dual of the initial lifts) for arbitrary U-sink.

1.2. Definition. A Prebornological Space is a pair (A, F) where F is a family of subsets of A that is closed under finite union and contains all finite nonempty subsets of A. See [4]p. 530. Furthermore, if $F \neq \varnothing$ and F is hereditary closed, then (A, F) is called a Bornological Space [4] p. 530 or [6] p. 1376. A morphism $(A, F) \rightarrow (A_1, F_1)$

of such spaces is a function f: $A \to A_1$ such that $f(C) \in F_1$ if $C \in F$. We denote by P Born and Born respectively, the categories so formed and by P Born*, the full subcategory of P Born determined by those spaces (A, F) with $\varnothing \notin F$. [4] p. 530. The categories PBorn, P Born*, and Born are topological over sets. See [4] p. 530.

1.3. The discrete structure (A, F) on A in (P Born, P Born*), Born is the set of all (nonempty) finite subsets of A. See [4]. p. 530.

1.4. A source $\{f_i: (A, F) \to (A_i, F_i)\ i \in I\}$ is initial in P Born, P Born*, and Born iff $F = \{B / B \subset A, f_iB \in F_i$ for all $i\}$. See [4] p. 530.

1.5. An epi morphism f: $(A_1, F_1) \to (A, F)$ is final in P Born or PBorn* (resp. Born) iff $F = \{f(B) / B \in F_1\}$ (resp. $F = \{B / B \subset A$ and $B \subset f(C)$ for some $C \in F_1\}$).

An epi sink $\{i_i, i_2: (A, F) \to (A_1, F_1)\}$ is final in P Born or PBorn* (resp. Born) iff $F_1 = \{B / B \subset A_1$ and B is (resp. contained in) a finite union of sets of the form $i_k(C)$ with $C \in F$, $k = 1, 2\}$. See [4] p. 530.

1.6. Lemma, Suppose f: $X \to Y$ is a morphism in P Born, PBorn*, or Born. If f has finite fibers i.e. $f^{-1}(y)$ is a finite set for all y in Y, then f reflects discreteness i.e. if Y is discrete, then so is X.

Proof: See [1] p. 6.

Let X be a set and $X^2 = X \times X$ be the cartesian product of X with itself. $X^2 V_\Delta X^2$ (two distinct copies of X^2 identified along the diagonal). A point (x, y) in $X^2 V_\Delta X^2$ will be denoted by$(x, y)_1$ ($(x, y)_2$) if (x, y) is in the first (resp. second) component of $X^2 V_\Delta X^2$. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$. [2] p. 3.

1.7. Definitions. The principal axis map, $A: X^2 V_\Delta X^2 \to X^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map, $S: X^2 V_\Delta X^2 \to X^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map $\nabla: X^2 V_\Delta X^2 \to X^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$.

Let $U: E \to$ Sets be topological and X an object in E with $UX = B$.

1.8. Definitions.

1. X is $\overline{T}_0$ iff the initial lift of the U-source $\{A: B^2 V_\Delta B^2 \to U(X^3) = B^3$ and $\nabla: B^2 V_D B^2 \to UD(B^2) = B^2\}$ is discrete.

2. X is T'_0 iff the initial lift of the U–source $\{id: B^2V_\Delta B^2 \to U(B^2V_\Delta B^2)' = B^2V_\Delta B^2$ and $\nabla: B^2V_\Delta B^2 \to U\ D\ (B^2) = B^2\}$ is discrete, where $(B^2V_\Delta B^2)'$ is the final lift of the U–sink $\{i_1, i_2: U(X^2) = B^2 \to B^2V_\Delta B^2\}$

3. X is T_1 iff the initial lift of the U–source $\{S: B^2V_\Delta B^2 \to U(X^3) = B^3$ and $\nabla: B^2V_\Delta B^2 \to U(D(B^2) = B^2\}$ is discrete.

4. X is $\mathrm{Pre}\overline{T}_2$ iff the initial lift of the U–sources $A: B^2V_\Delta B^2 \to U(X^3) = B^3$ and $S: B^2V_\Delta B^2 \to U(X^3)$ agree.

5. X is $\mathrm{Pre}\overline{T}'_2$ iff the inital lift of the U–source $S: B^2V_\Delta B^2 \to U(X^3)$ and the final lift of the U–sink $i_1, i_2: U(X^2) \to B_2V_\Delta B^2$ agree.

6. X is $\overline{T}_2$ iff X is $\overline{T}_0$ and $\mathrm{Pre}\overline{T}_2$.

7. X is T'_2 iff X is T'_0 and $\mathrm{Pre}T'_2$.

8. X is ΔT_2 iff the diagonal, Δ, is closed in X^2. See [1] p. 8.

9. X is ST_2 iff the diagonal, Δ, is strongly closed in X^2. See [1] p. 8.

1.9. Remark. We define π_{ij} by $\pi_i + \pi_j: B^2V_\Delta B^2 \to B$, where $\pi_i: B_2 \to B$ is the ith projection $i = 1, 2$. Note that $\pi_1 A = \pi_{11} = \pi_1 S$, $\pi_2 A = \pi_{21} = \pi_2 S$, $\pi_3 A = \pi_{12}$ and $\pi_3 S = \pi_{22}$. When showing that A and S are initial, it is sufficient to show that (π_{11}, π_{21} and π_{12}), and (π_{11}, π_{21} and π_{22}) are initial lifts, respectively. See [2] p. 13.

2. Separation Properties

In this section, we give explicit characterizations of the generalized separation properties for the topological categories of P Born, P Born*, and Born.

2.1. Lemma. If $\nabla: (B^2V_\Delta B^2, K) \to (B^2, K_d)$ is in any one of P Born, P Born*, or Born, where K_d is discrete structure on B, then K is discrete.

Proof: This follows from 1.6 since the fibers of ∇ are finite.

2.2. Theorem. All objects in P Born, P Born* or Born are T'_0, $\overline{T}_0$, and T_1.

Proof: This follows from 2.1 and Definition 1.8.

2.3. Theorem. $X = (B, F)$ in P Born or P Born* is $\mathrm{Pre}\overline{T}_2$ iff X is strictly hereditary closed i.e. if $\varnothing \neq V \subset U$ and $U \in F$, then $V \in F$.

Proof: Suppose X is $\mathrm{Pre}\overline{T}_2$ i.e. by 1.4 and 1.9 for any subset W of the wedge if $\pi_{11}\ W \in F$ and $\pi_{21}\ W \in F$, then $\pi_{12} W \in F$ iff $\pi_{22} W \in F$. We must show that if $\varnothing \neq V \subset U$, then $V \in F$ if $U \in F$. If $V = U$, then clearly $V \in F$. If $\varnothing \neq V \neq U$ and $V \subset U$, then let $W = (V \times U) \vee (U - V \times V)$ and note that $\pi_{11} W = U \in F$, $\pi_{21} W = U \in F$, $\pi_{22} W = U \cup V = U \in F$, and $\pi_{12} W = V \cup V = V$. Since X is $\mathrm{Pre}\overline{T}_2$, it follows that $\pi_{12} W = V \in F$.

Conversely, we shall show that if X is strictly hereditary closed, then X is $\mathrm{Pre}\overline{T}_2$ i.e. if $W = U \vee V$ is any subset of the wedge with $\pi_{11} W = \pi_1 U \cup \pi_1 V \in F$, $\pi_{21}\ W = \pi_2\ U \cup \pi_1\ V \in F$, then $\pi_{12} W = \pi_1 U \cup \pi_2\ W \in F$ iff $\pi_{22} W = \pi_2 U \cup \pi_2 V \in F$. To this end, assume π_{11} W and π_{21} W are in F. If $U = \varnothing \neq V$, then $\pi_{12} = \pi_2 V \in F$ iff $\pi_{22} W = \pi_2 V \in F$. If $U \neq \varnothing = V$, then $\pi_{11} W = \pi_1 U \in F$, $\pi_{21} W = \pi_2 U \in F$ and consequently $\pi_{12} W = \pi_1 U \in F$ iff $\pi_{22} W = \pi_2 U \in F$. If $U \neq \varnothing \neq V$, then $\pi_{11} W = \pi_1 U \cup \pi_1 V \in F$ and $\pi_{21} W = \pi_2 U \cup \pi_1 V \in F$ imply by assumption that $\pi_1 U$, $\pi_2 U$, $\pi_1 V \in F$ and consequently, $\pi_{12}\ W = \pi_1 U \cup \pi_2 V \in F$ iff $\pi_{22} W = \pi_2 U \cup \pi_2 V \in F$. If X is in P Born and $U = \varnothing = V$, then $W = \varnothing$ and if $\pi_{11} W = \varnothing = \pi_{21}\ W \in F$, then $\pi_{12} W = \varnothing \in F$ iff $\pi_{22} W = \varnothing \in F$. This completes the proof.

2.4. Theorem. $X = (B, F)$ in P Born or P Born* is PreT'_2 iff X is hereditary closed.

Proof: Suppose X is PreT'_2 i.e. by 1.4, 1.9, and 1.5 for any subset W of the wedge (a) $\pi_{11} W \in F$, $\pi_{21} W \in F$, and $\pi_{22} W \in F$ iff (b) $W = i_1 W_1 \cup i_2 W_2$ for some $W_1, W_2 \in F^2$ where F^2 is defined by $N \in F^2$ iff $\pi_1 N \in F$ and $\pi_2 N \in F$. We will show that if $U \in F$ and $V \subset U$, then $V \in F$. If $V = U$, then $V \in F$. If $V \neq U$ and $V \subset U$, then let $W = V^2 \vee (U - V)^2$ and clearly $\pi_{11}\ W = U = \pi_{21} W = \pi_{22} W \in F$. Since X is PreT'_2, it follows that $W = i_1 W_1 \cup i_2 W_2$ and consequently $W_1 = V^2 \in F^2$. Thus, $\pi_1 W_1 = V \in F$.

Conversely, suppose X is hereditary closed. We will show that X is PreT'_2 i.e. (a) and (b) are equivalent. To show (b) implies (a) note that if $W = i_1 W_1 \cup i_2 W_2$, then clearly $\pi_{11} W = \pi_1 W_1 \cup \pi_1 W_2 \in F$, $\pi_{21} W = \pi_2 W_1 \cup \pi_1 W_2 \in F$, and $\pi_{22} W = \pi_2 W_1 \cup \pi_2 W_2 \in F$ (since W_1 and W_2 are in F^2 iff $\pi_1 W_1 \in F$ and $\pi_2 W_1 \in F$, and $\pi_1 W_2$

$\in$ F and $\pi_2 W_2 \in F$). On the other hand, if $W = U \vee V$, where U, V are subsets of B^2, and $\pi_{11} W = \pi_1 U \cup \pi_1 V \in F$, $\pi_{21} W = \pi_2 U \cup \pi_1 V \in F$, and $\pi_{22} W = \pi_2 U \cup \pi_2 V \in F$, then, by assumption $\pi_i U$ and $\pi_i V$ are in F for all i = 1 2. and consequently, U, V are in F^2. Clearly, $W = i_1 U \vee i_2 V$ and thus (a) implies (b). Therefore (a) and (b) are equivalent i.e. X is $PreT'_2$.

2.5. Theorem. X = (B, F) in P Born or P Born* is $\overline{T}_2$ iff X is strictly hereditary closed i.e. if $\varnothing \neq V \subset U$ and $U \in F$, then $V \in F$.

Proof: Combine 2.2, 2.3, and Definition 1.8.

2.6. Theorem. X = (B, F) in P Born or P Born* is T'_2 iff X is herediatary closed.

Proof: Combine 2.2, 2.4, and Definition 1.8.

2.7. Remark. In P Born and P Born*, $PreT'_2$ and T'_2 imply $Pre\overline{T}_2$ and $\overline{T}_2$, respectively.

2.8. Theorem. Every object in Born is $Pre\overline{T}_2$, $PreT'_2$, $\overline{T}_2$, and T'_2.

Proof: This follows from the fact that X is heredidary closed.

2.9. Theorem. Let X = (B, F) be in P Born, P Born* or Born. X is $\triangle T_2$ iff $B = \varnothing$ or a point.

Proof: [1] p. 17.

2.10. Theorem. All X in P Born, P Born*, or Born are ST_2.

Proof: [1] p. 17.

2.11. Remark. Except for $\triangle T_2$, all of the other separation properties defined in 1.8 are equvialent in Born. Some of the "T_2" structures could be equal while others could be different. For example, in Born, T'_2, ST_2 and $\overline{T}_2$ are all equivalent and all are implied by but are different from $\triangle T_2$. In PBorn and P Born*, T'_2 and $\overline{T}_2$ are equivalent, are implied by $\triangle T_2$, and imply ST_2.

BORNOLOJİK VE PREBORNOLOJİK KATEGORİ UZAYLARINDA AYRILMA AKSİYONLARI

ÖZET

Bu çalışmada, Bornolojik uzaylar ve Prebornolojik uzaylarında T_0, T_i, Pre T_2 ve T_2 ayrılma özelliklerinin her birinin açık bir karakte-

rezisyonu verildi. Bundan başka, bu kategorilerde değişik T_0, Pre T_2 ve T_2 yapıları arasında ortaya çıkan özel ilişkiler incelendi.

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