Commun. Fac. Sci. Univ. Ank. Series A,
V. 42. pp. 45-53 (1993)

COMPATIBLE MAPPINGS OF TYPE (A) AND COMMON FIXED POINTS IN BANACH SPACES

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(Received Feb. 17, 1993; Accepted April 13, 1993)

ABSTRACT

In this paper, we introduce the concept of compatible mappings of type (A) on Banach spaces and give a common fixed point theorem for compatible mappings of type (A).

I. INTRODUCTION

In [3], M. Gregus Jr. proved the following theorem:

Theorem A. Let C be a closed convex subset of a Banach space X. If T is a mapping C into itself satisfying the following inequality:

$$(1.1) \qquad \|Tx - Ty\| < a\|x - y\| + b\|Tx - x\| + c\|Ty - y\|$$

for all x, $y \in C$, where $0 < a < 1$, $c \geq 0$, $b \geq c$ and $a + b + c = 1$, then T has a unique fixed point in C

Recall that the mapping T satisfying (1.1) with $a = 1$ and $b = c = 0$ is said to be non-expansive and this mapping satisfying (1.1) with $a = 0$ and $b = c = \frac{1}{2}$ was considered by C.S. Wong ([7]).

Recently, B. Fisher and S. Sessa ([2]), M.L. Diviccaro et al ([1]) and R.N. Mukherjee et al ([6]) generalized Theorem A in various ways.

For example, B. Fisher and S. Sessa ([2]) proved the following theorem:

Theorem B. Let T and I be mappings of a closed convex subset C of a Banach space X into itself satisfying the following conditions

$$(1.2) \qquad T(C) \subset I(C),$$

$$(1.3) \qquad \|ITx - TIx\| \leq \|Ix - Tx\|$$

for all $x \in X$

$$(1.4) \qquad \|Tx - Ty\| \leq a\|Ix - Yy\| + (1-a)\max\{\|Tx - Ix\|, \|Ty - Iy\|\}$$

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for all x, y $\in$ C, where $0 < a < 1$. Then if I is linear and non-expansive in C, then I and T have a unique common fixed point in C.

In this paper, we introduce the concept of compatible mappings of type (A) on Banach spaces, which is, of course, equivalent to the concept of compatible mappings which was introduced by G. Jungck ([4]) under some conditions, and give a common fixed point theorem for compatible mappings of type (A) satisfying the contraction condition of Gregus type.

II. COMPATIBLE MAPPINGS OF TYPE (A)

In [5], G. Jungck, P.P. Murthy and T.J. Cho proved that two pairs of compatible mappings and compatible mapping of type (A) on a metric space are equivalent to each other under some conditions. Now, in this section, we give some properties of compatible mappings and compatible mappings of type (A) on Banach spaces:

Definition 2.1. Let S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself. A pair $\{S, T\}$ is said to be compatible on X if

$$\|STx_n - TSx_n\| \to 0 \text{ as } n \to \infty$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Definition 2.2. Let S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself. A pair $\{S, T\}$ is said to be compatible of type (A) on X if

$$\|TSx_n - SSx_n\| \to 0 \text{ and } \|STx_n - TTx_n\| \to 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

The proofs of the following propositions follow from the same lines in [5]:

Proposition 2.1. Let S and T be continuous mappings from a Banach space $(X, \|\cdot\|)$ into itself. If a pair $\{S, T\}$ is compatible on X, then it is compatible of type (A) on X.

Proposition 2.2. Let S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself and let a pair $\{S, T\}$ be compatible of type (A) on X. If one of S and T is continuous, then the pair $\{S, T\}$ is compatible on X.

The following proposition is a direct consequence of Propositions 2.1 and 2.2:

Proposition 2.3. Let S and T be in Proposition 2.1. Then a pair $\{S, T\}$ is compatible on X if and only if it is compatible of type (A) on X.

Now, we give two examples to illustrate Proposition 2.3:

Example 2.1. Let $X = \mathbf{R}$ with the Euclidean norm $\|\ .\ \|$ and define two mappings S and $T : X \to X$ as follows:

$$Sx = \frac{x}{4} \text{ and } Tx = \frac{x}{2}$$

for all $x \in X$. Both S and T are continuous at $x = 0$ and $S(0) = T(0)$.

Now, consider a sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{2^n}$, $n = 1$, 2, ..., Then we have

$$Sx_n = \frac{1}{2^{n+2}} \to 0 = z \text{ and } Tx_n = \frac{1}{2^{n+1}} \to 0 = z \text{ as } n \to \infty,$$

that is, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0 = z$, and the pair $\{S, T\}$ is compatible of type (A) on X since we have

$$\|STx_n - TTx_n\| = \left| \frac{1}{2^{n+3}} - \frac{1}{2^{n+2}} \right| = \frac{1}{2^{n+3}} \to 0 \text{ as } n \to \infty$$

and

$$\|TSx_n - SSx_n\| = \left| \frac{1}{2^{n+3}} - \frac{1}{2^{n+4}} \right| = \frac{1}{2^{n+4}} \to 0 \text{ as } n \to \infty$$

Further, pair $\{S, T\}$ is compatible on X. In fact.

$$\|STx_n - TSx_n\| = \left| \frac{1}{2^{n+3}} - \frac{1}{2^{n+3}} \right| = 0$$

The following example shows that Proposition 2.3 is not true if S and T are not continuous at a point in X.

Example 2.2. Let $X = \mathbf{R}$ with the Euclidean norm $\|\ .\ \|$ and define two mapping S and $T : X \to X$ as follows:

$$Sx = \begin{cases} \frac{1}{x^3} & \text{if } x \neq 0, \\ \frac{1}{2} & \text{if } x = 0, \end{cases} \quad \text{and } Tx = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ \frac{1}{5} & \text{if } x = 0. \end{cases}$$

Both S and T are discontinuous at $z = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n^2$, $n = 1,2,3,\ldots$. Then $\lim_{1\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0 = z$. Since

$$\|STx_n - TSx_n\| = |n^6 - n^6| = 0,$$

the pair $\{S, T\}$ is compatible on X. But

$$\|STx_n - TTx_n\| = |n^6 - n^4| = n^4 \, |n^2 - 1| \to \infty \text{ as } n \to \infty$$

implies that the pair $\{S, T\}$ is not compatible of type (A) on X.

We need the following propositions for our main theorems. The proofs of the following propositions also follow from the same lines in [5]:

Proposition 2.4. Let S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself. If a pair $\{S, T\}$ is compatible of type (A) on X and $Sz = Tz$ for some $z \in X$, Then $STz = TTz = TSz = SSz$.

Proposition 2.5. Let S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself. Let a pair $\{S, T\}$ be compatible of type (A) on X and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Then we have

(1) $\|TSx_n - Sz\| \to 0$ as $n \to \infty$ if S is continous.

(2) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

III. COMMON FIXED POINTS

Let A, B, S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself such that

(3.1) $\quad A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(3.2) $\quad \|Ax - By\| \leq a\|Sx - Ty\| + b \max\{\|Ax - Sx\|, \|By - Ty\|, \frac{1}{2}(\|Ax - Ty\| + \|By - Sx\|)\}$

for all $x, y \in X$, where $a, b > 0$ and $a + b < 1$. Then, by (3.1), since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

(3.3) $$y_{2n} = Sx_{2n} = Bx_{2n-1} \text{ and } y_{2n+1} = Tx_{2n+1} = Ax_{2n}$$

for $n = 0,1,2,\ldots$

Then we have the following lemma for our main theorem:

Lemma 3.1. Let A, B, S and T be mappings from a Banach space $(X, \|\cdot\|)$ into itself satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.

Proof: By (3.2), we have

(3.4) $$\begin{aligned} \|y_{2n+1} - y_{2n}\| &= \|Ax_{2n} - Bx_{2n-1}\| \\ &\leq a\|Sx_{2n} - Tx_{2n-1}\| + b \max\{\|Sx_{2n} - Ax_{2n}\|, \\ &\quad \|Tx_{2n-1} - Bx_{2n-1}\|, \tfrac{1}{2}(\|Ax_{2n} - Tx_{2n-1}\| + \|Bx_{2n-1} - Sx_{2n}\|)\} \\ &= a\|y_{2n} - y_{2n-1}\| + b \max\{\|y_{2n} - y_{2n+1}\|, \\ &\quad \|y_{2n-1} - y_{2n}\|, \tfrac{1}{2}(\|y_{2n+1} - y_{2n}\| + \|y_{2n} - y_{2n-1}\|)\}. \end{aligned}$$

If $\|y_{2n+1} - y_{2n}\| > \|y_{2n} - y_{2n-1}\|$ in 3.4), then we have

$$\begin{aligned} \|y_{2n+1} - y_{2n}\| &\leq (a+b)\|y_{2n+1} - y_{2n}\| \\ &< \|y_{2n+1} - y_{2n}\|, \end{aligned}$$

which is a contraction since $a + b < 1$ and so

$$\|y_{2n+1} - y_{2n}\| \leq (a+b)\|y_{2n} - y_{2n-1}\|.$$

Similarly, we have

$$\|y_{2n} - y_{2n-1}\| \leq (a+b)\|y_{2n-1} - y_{2n-2}\|.$$

Therefore, we have

(3.5) $$\begin{aligned} \|y_{n+1} - y_n\| &\leq (a+b)\|y_n - y_{n-1}\| \\ &\cdots\cdots\cdots\cdots \\ &\leq (a+b)^n \|y_1 - y_0\|. \end{aligned}$$

If $m \geq n$, then the repeated use of (3.5) yields

$$\begin{aligned} \|y_m - y_n\| &\leq \|y_m - y_{m-1}\| + \|y_{m-1} - y_{m-2}\| + \cdots + \|y_{n+1} - y_n\| \\ &\leq \{(a+b)^{m-2} + (a+b)^{m-3} + \ldots + (a+b)^{n-1}\}\|y_1 - y_0\| \\ &= \frac{(a+b)^{n-1}}{1-(a+b)}\|y_1 - y_0\|. \end{aligned}$$

Therefore, since $0 < a + b < 1$, the sequence $\{y_n\}$ is a Cauchy sequence in X.

Now, we are ready to give our main theorem:

Theorem 3.2. Let A, B, S and T be mappings from a Banach space $(X, \|.\|)$ into itself satisfying the conditions (3.1), (3.2), (3.6) and (3.7):

(3.6) one of A, B, S and T is continuous,

(3.7) the pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (A) on X.

Then A, B, S and T have a unique common fixed point in X.

Proof: By Lemma 3.1, the sequence $\{y_n\}$ is defined by (3.3) is a Cauchy sequence in X. Since $(X, \|.\|)$ is a Banach space, $\{y_n\}$ converges to a point z in X. The subsequences $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that T is continuous. Then we have

$$TTx_{2n-1},\ TBx_{2n-1} \to Tz \text{ as } n \to \infty.$$

Since $\{B, T\}$ is compatible of type (A) on X, by Proposition 2.5,

$$BTx_{2n-1} \to Tz \text{ as } n \to \infty.$$

Then, by (3.2), we have

(3.8) $$\|Ax_{2n} - BTx_{2n-1}\| < a\|Sx_{2n} - TTx_{2n-1}\| + b \max\{\|Sx_{2n} - Ax_{2n}\|, \|TTx_{2n-1} - BTx_{2n-1}\|, \tfrac{1}{2}(\|Ax_{2n} - TTx_{2n-1}\| + \|BTx_{2n-1} - Sx_{2n}\|)\}.$$

Taking $n \to \infty$ in (3.8), we have

$$\begin{aligned}\|z - Tz\| &\leq a\|z - Tz\| + b \max\{0, 0, \|z - Tz\|\}\\ &= (a + b)\|z - Tz\|\\ &< \|z - Tz\|,\end{aligned}$$

which is a contradiction and so $Tz = z$. Again from (3.2), we also have

(3.9) $$\|Ax_{2n} - Bz\| \leq a\|Sx_{2n} - Tz\| + b \max\{\|Ax_{2n} - Sx_{2n}\| \|Bz - z\|, \tfrac{1}{2}(\|Ax_{2n} - Tz\| + \|Bz - Sx_{2n}\|)\}.$$

Taking $n \to \infty$ in (3.9), we get

$$\begin{aligned}\|z - Bz\| &\leq a\|z - Tz\| + b \max\{0, \|z - Bz\|, \tfrac{1}{2}\|z - Bz\|\}\\ &= b\|z - Bz\|\\ &< \|z - Bz\|,\end{aligned}$$

which is a contradiction and so $Bz = z$. Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Bz = Su$.

Now, we claim that Au = z. If Au $\neq$ z, then, by (3.2), we have

$$\begin{aligned}\|Au - z\| &= \|Au - Bz\| \\ &\leq a\|Su - Tz\| + b \max\{\|Au - Su\|, \\ &\|Bz - Tz\|, \tfrac{1}{2}(\|Au - Tz\| + \|Bz - Su\|)\} \\ &= < b\|Au - z\| \\ &< \|Au - z\|,\end{aligned}$$

which is also a contradiction and so Au = z = Su. Since pair {A, S} is compatible of type (A) on X, by Proposition 2.4, ASu = SSu, that is Az = ASu = SSu = Sz.

Finally, we claim that Az = z. If Az $\neq$ z, then, by (3.2) again, we have

$$\begin{aligned}\|Az - z\| &= \|Az - Bz\| \\ &\leq a\|Sz - Tz\| + b \max\{\|Az - Sz\|, \\ &\|Bz - Tz\|, \tfrac{1}{2}(\|Az - Tz\| + \|Bz - Sz\|)\} \\ &\leq (a + b)\|Az - z\| \\ &< \|Az - z\|,\end{aligned}$$

which is a contradiction and so Az = z. Therefore, z is a common fixed point of A, B, S and T. The uniqueneess of a common fixed point z follows easily from (3.2). Similarly, we can complete the proof when A or B or S is continuous. This completes the proof.

As an immediate consequence of Theorem 3.2, we have the following:

Corollary 3.3. Let A, B, S and T be mappings from a Banach space (X, $\|\cdot\|$) into itself satisfying the conditions (3.1), (3.6), (3.7) and (3.10):

(3.10) $\|Ax - By\| < a\|Sx - Ty\| + b\max\{\|Ax - Sx\|, \|By - Ty\|\}$
for all x, y $\in$ X, where a, b $>$ 0 and a + b $<$ 1.

Then A, B, S and T have a unique common fixed point in X.

The following example illustrate our main theorem:

Example 3.1. Let X = $\{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$ with the Euclidean norm $\|\cdot\|$ and define A, B, S and T : X $\to$ X by

$$A(0) = B(0) = S(0) = T(0) = 0,$$

$$A\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2^{n+4}} & \text{if n is even,} \\ \frac{1}{2^{n+5}} & \text{if n is odd,} \end{cases} \quad B\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2^{n+5}} & \text{if n is even,} \\ \frac{1}{2^{n+8}} & \text{if n is odd,} \end{cases}$$

$$S\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2^{n+7}} & \text{if n is even,} \\ \frac{1}{2^{n+4}} & \text{if n is odd,} \end{cases} \quad T\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2^{n+4}} & \text{if n is even,} \\ \frac{1}{2^{n+3}} & \text{if n is odd.} \end{cases}$$

Then we have the following:

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

(2) One of A, B, S and T is continuous.

(3) If $x = \frac{1}{2}$ and $y = \frac{1}{2^2}$ in (3.2), then we have

$$\left|\frac{1}{2^6} - \frac{1}{2^7}\right| \leq a \left|\frac{1}{2^5} - \frac{1}{2^6}\right| +$$

$$b\max\left\{\left|\frac{1}{2^6} - \frac{1}{2^5}\right|, \left|\frac{1}{2^7} - \frac{1}{2^6}\right| \frac{1}{2}\left(\left|\frac{1}{2^6} - \frac{1}{2^6}\right| + \left|\frac{1}{2^7} - \frac{1}{2^5}\right|\right)\right\}$$

$$= \frac{a}{2^6} + b\max\left\{\frac{1}{2^6}, \frac{1}{2^7}, \frac{3}{2^8}\right\}$$

$$= \frac{1}{2^6}(a + b)$$

or equivalently,

$$\frac{1}{2^7} \leq \frac{1}{2^6}(a + b)$$

$$1 \leq 2(a + b)$$

Thus, if $a = \frac{1}{4}$ and $b = \frac{1}{2}$, then $1 < \frac{3}{2}$ and so the condition (3.2) holds.

(4) The pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (A) on X, but $AS \neq SA$, $BT \neq TB$, $AB \neq BA$ and $ST \neq TS$. Moreover, for

$$x_n = \frac{1}{2^n}, \; n = 0, 1, 2, \ldots,$$

$$Ax_{2n} = Tx_{2n+1} = \frac{1}{2^{2n+4}} \text{ and } Bx_{2n+1} = Sx_{2n+2} = \frac{1}{2^{2n+9}}$$

Thus, the sequence $\{y_n\}$ defined by

$$y_{2n} = Sx_{2n} = Bx_{2n-1} \text{ and } y_{2n+1} = Tx_{2n+1} = Ax_{2n}$$

converge to 0, which is the only common fixed point of A, B, S and T.

Remark. (1) If $b = 0$, $A = B = F$ and $S = T = I_X$ (the identity mappint on X) in (3.2), then we have

$$\|Fx - Fy\| \leq a \; \|x - y\|$$

for all $x, y \in X$, where $0 < a < 1$, that is, F is a contraction mapping on X. Hence, Theorem 3.2 generalizes the Banach's Fixed Point Theorem.

(2) If $b = 0$, $a = 1$, $A = B = F$ and $S = T = I_X$ in (3.2), then we have

$$\|Fx - Fy\| < \|x - y\|$$

for all $x, y \in X$, that is, F is a non-expansive mapping on X.

Acknowledgement: The authors wish to express the deepest appreciation to Prof. S. Sessa for his helpful comment on this paper.

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