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SOME CLOSURE THEOREMS FOR THE SPACE G(λ)

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ABSTRACT

Using the notion of uniqueness set of entire functions, a closure theorem for the space G(λ) of entire functions is obtained. As a consequence, two more closure theorems are established corresponding to the uniqueness theorems in Boas [1].

1. INTRODUCTION

The object of this short note is to prove three closure theorems for the Hilbert space G(λ) of entire functions. Using the notion of uniqueness sets of entire functions. Describing the necessary preliminaries in § 2, we shall establish the closure theorems for the Hilbert space G(λ) in § 3.

2. Let us consider the class of all power series $f(z) = \Sigma a_n z^n$ such that $\Sigma \lambda_n |a_n|^2 < \infty$ where $\lambda_n > 0$, $n = 0,1,2,3,\ldots$ and $(\lambda_n)^{1/n} \to 0$ as $n \to \infty$. We can easily verify that f(z) is an entire function. Let G(λ) denote the class of all such entire functions defined in this way. G(λ) becomes a Hilbert space with the inner product $(f,g) = \Sigma \lambda_n a_n \overline{b_n}$. The basic properties of such a Hilbert space were studied by the author [2].

Definition: Let f be an entire function. A sequence (z_n) of complex numbers is called a uniqueness set for f if

$$f(z_n) = 0 \text{ for } n = 1,2,3, \ldots \text{ implies } f(z) \equiv 0.$$

Let E be a subset of G(λ) .Let L(E) stand for the closed linear subspace of G(λ) generated by the elements of E. Since G(λ) is a Hilbert space, the space of all continuous linear functionals coincide with G(λ) itself by Riesz representation theorem. We shall make use of the following fundamental theorem which we quote as Lemma in proving the closure theorems for the space G(λ) of entire functions.

Lemma: Let $\varnothing$ be a continuous linear functional on $G(\lambda)$. Then any element α in $G(\lambda)$ belongs to $L(E)$ if and only if every continuous functional $\varnothing$ which vanishes for all $\beta \in E$ vanishes identically on $G(\lambda)$.

3.1. Let $L(\lambda)$ be the set of all entire functions f such that $|f(z)| \leq c\sqrt{D(z)}$ where c is a constant and $D(z) = \Sigma \dfrac{z^{2n}}{\lambda_n}$. It is easy to check that $L(\lambda)$ is a linear space and $G(\lambda)$ is a linear subspace of $L(\lambda)$. Let (z_n) be a sequence of complex numbers such that it is a uniqueness set for the class $L(\lambda)$ in the sense that $\alpha(z_n) = 0$ for $n = 1,2,3,\ldots$, then $\alpha(z) \equiv 0$ for $\alpha \in L(\lambda)$. Using these we shall establish the following closure theorem for $G(\lambda)$.

Theorem 1: Let $\sum_{p=0}^{\infty} a_p z^p \in G(\lambda)$ be such that no a_p is zero. If (z_n) is a uniqueness set as defined above, then

$$L\{\alpha_n : n > 1\} = G(\lambda) \text{ where}$$

$$\alpha_n = \alpha(z.z_n) = \sum_{p=0}^{\infty} a_p z_n^p z^p.$$

Proof: Let $\varnothing$ be a continuous linear functional on $G(\lambda)$ and let it be uniquely determined by $F = \sum_0^{\infty} b_p z^p \in G(\lambda)$. Then $\varnothing(\alpha_n) = (\alpha_n, \varnothing) = \Sigma \lambda_p a_p z_n^p \overline{b_p}$.

Let us consider $g(z) = \Sigma \lambda_p a_p \overline{b_p} z^p$.

Since $(\overline{b}_p)$ is bounded and $\Sigma \lambda_p |a_p|^2$ is convergent, we have by Hölder's inequality,

$$|g(z)|^2 \leq K \left\{ \Sigma \frac{|z|^{2n}}{\lambda_n} \right\}.$$

Therefore $g(z)$ belongs to the linear class $L(\lambda)$. Since (z_n) is a uniqueness set for $g(z)$ in $L(\lambda)$, we have $g(z) \equiv 0$. Therefore $\lambda_p a_p \overline{b_p} = 0$. Since $\lambda_p a_p \neq 0$, $b_p = 0$. Hence $\varnothing$ is an identically zero functional on $G(\lambda)$. So by the Lemma in § 2, we get $L(\alpha_n : n \geq 1) = G(\lambda)$. This completes the proof of the theorem.

3.2. Let $f(u) = \sum_{p=0}^{\infty} \frac{u^p}{p}$ where $u = |z|^2$ and $f(u)$ is assumed to be of growth (ρ, σ). Since $\sqrt{f(u)}$ dominates the functions of the class $G(\lambda)$, the functions of the class $G(\lambda)$ are of growth $\left(2\rho, \frac{\sigma}{2}\right)$. We shall use the notation of Boas [1] in the following two theorems. Now corresponding to the uniqueness theorems of Boas [1, pp 152–153], we have the following closure theorems.

Theorem 2: Let $\alpha = \sum_{p=0}^{\infty} a_p z^p$ of growth $\left(2\rho, \frac{\sigma}{2}\right)$ belongs to the class $G(\lambda)$ with no $a_p = 0$ for $p = 0,1,2,\ldots$, $\alpha(z_n) = 0$ for a sequence of complex numbers (z_n) If ρ and σ satisfy one or other f the following conditions,

(a) $\sigma \leq \frac{1}{\rho} \liminf_{r \to \infty} \frac{n_a(r)}{r^2}$

(b) $\sigma \leq \frac{1}{\rho e} \limsup_{r \to \infty} \frac{n_a(r)}{r^2}$

then $L \{\alpha_n = \alpha(z.z_n),\ n = 1,2,3,\ldots\} = G(\lambda)$.

Proof: Under the conditions states in the theorem, (z_n) is a uniqueness set for the class $L(\lambda)$ of growth (ρ,σ). Therefore by Theorem 1, we get the required result.

3.3. When $f(u)$ is of growth $\left(\frac{1}{2}, \sigma\right)$, then the functions of the class $G(\lambda)$ are of exponential type. They are of growth $\left(1, \frac{\sigma}{2}\right)$.

Making use of the another uniqueness theorem of Boas [1, p. 154], we have the following closure theorem.

Theorem 3: Let $\alpha = \Sigma\, a_p z^p$ be of exponential type $\sigma/2$ and belong to $G(\lambda)$ with no $a_p = 0$ for $p = 0,1,2,3,\ldots$, Let (z_n) be a sequence of complex numbers such that $\alpha(z_n) = 0$. If σ satisfies one or other of the following two conditions,

(i) $\frac{\sigma}{2} \leq \liminf_{r\to\infty} \frac{n_a(r)}{r}$

(ii) $\frac{\sigma}{2} \leq \limsup_{r\to\infty} \frac{n_a(r)}{e^r}$

then $L\{\alpha_n = \alpha(z,z_n),\ n = 1,2,3,\ldots\} = G(\lambda)$.

Proof: Using the conditions of the theorem (z_n) becomes a uniqueness set for the class $L(\lambda)$ of growth $(1, \sigma/2)$. Hence Theorem 1 yields the given result.

REFERENCES

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