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INVOLUTE - EVOLUTE CURVE COUPLES OF HIGHER ORDER IN $\mathbb{R}^n$ AND THEIR HORIZONTAL LIFTS IN $T\mathbb{R}^n$

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ABSTRACT

The evolute of a given curve is a well-known concept in 3-dimensional Euclidean space $\mathbb{R}^3$. In this paper, we generalize this concept to n-dimensional Euclidean space $\mathbb{R}^n$. In addition, defining an involute-evolute curve couple (α, β) in $\mathbb{R}^n$, we obtain some properties connected with (α, β) and its horizontal lift (α^H, β^H) in $T\mathbb{R}^n$.

INTRODUCTION

Let $\{V_1, V_2, \ldots, V_n\}$ be the Frenet frame field of a curve $x: I \to \mathbb{R}^n$, $I \subset \mathbb{R}$, where $V_1 = \dot{x}(s)$ for eachs. Then, for $h < n$, the vector fields $V_1, V_2, \ldots, V_h$ span an h-dimensional space called the h-dimensoinal osculating space of x (Gerretsen, 1962).

Let $x: I \to \mathbb{R}^n$ be a given curve in $\mathbb{R}^n$ with the Frenet frame field $\{V_1, V_2, \ldots, V_n\}$ and consider the curve

$$y(s) = x(s) + \sum_{i=1}^{k} \xi_i V_i, \; \xi_i = \xi_i(s), \; k < n-1. \tag{1}$$

Those curves (1) which are orthogonal trajectories of the system of the k-dimensional osculating spaces of x are called involutes of order k of the given curve x (Gerretsen, 1962).

Now let

$$g = \sum_{i,j=1}^{n} g_{ji} \, dx^j \circ dx^i : (g_{ji})$$

be a Riemannian metric on $\mathbb{R}^n$ and let ∇ be the Riemannian connection of $\mathbb{R}^n$. Then the vertical lift g^V and the horizontal lift g^H of g have respectively the components

$$g^V: \begin{bmatrix} g_{ji} & 0 \\ 0 & 0 \end{bmatrix}, \quad g^H: \begin{bmatrix} \sum_{h=1}^{n} (\Gamma_j^h g_{hi} + \Gamma_i^h g_{jh} & g_{ji} \\ g_{ji} & 0 \end{bmatrix}$$

in the induced coordinate system (x^h, y^h) in the tangent manifold $T\mathbb{R}^n$,

where $\Gamma_j^h = \sum_{r=1}^{n} y^r \Gamma^h{}_{rj}$ and (x^h) is a system of local coordinates

defined in neighborhood U in $\mathbb{R}^n$ (Yano and Ishiara, 1973).

Let us set $g^{VH} = g^V + g^H$. Thus the metric g^{VH} has the components

$$g^{VH}: \begin{bmatrix} g_{ji} + \sum_{h=1}^{n} (\Gamma_j^h g^h{}_i + \Gamma_i^h g_j{}^h) & g_{ji} \\ g_{ji} & 0 \end{bmatrix} \tag{2}$$

and it is a pseudo–Riemannian metric on $T\mathbb{R}^n$.

Let there be given a vector field $X = \sum_{h=1}^{n} X^h \frac{\partial}{\partial x^h}$ on $\mathbb{R}^n$.

Then we define the horizontal lift X^H of X by

$$X^H: \begin{bmatrix} X^h \\ -\sum_{i=1}^{n} \Gamma_i^h X^i \end{bmatrix} \tag{3}$$

with respect to the induced coordinates in $T\mathbb{R}^n$ (Yano and Ishiara, 1973). The horizontal lift X^H of X to $T\mathbb{R}^n$ is a projectable vector field with projection X. Furthermore the horizontal lift X^H of X is a horizontal vector field.

We denote by ∇^H the horizontal lift of ∇ to $T\mathbb{R}^n$ and by $\pi: T\mathbb{R}^n \to \mathbb{R}^n$ the projection mapping.

INVOLUTE–EVOLUTE CURVE COUPLES

Definition 1: Let $\alpha, \beta: I \to T\mathbb{R}^n$, $I \subset \mathbb{R}$, be two curves in $T\mathbb{R}^n$. Then the curve couple (α,β) in $\mathbb{R}^n$ such that

$$(\alpha, \beta) = \pi o\ (\tilde{\alpha}, \tilde{\beta}) = (\pi o\tilde{\alpha}, \pi o\tilde{\beta}) \tag{4}$$

is called the projection of the couple $(\tilde{\alpha}, \tilde{\beta})$. In this case, we say that $(\tilde{\alpha}, \tilde{\beta})$ is projectable with projection (α, β).

Definition 2: A curve couple $(\tilde{\alpha}, \tilde{\beta})$ in $T\mathbb{R}^n$ is horizontal if the tangent vectors of $\tilde{\alpha}$ and $\tilde{\beta}$ are horizontal at each point. Given a curve couple (α, β) in $\mathbb{R}^n$, a horizontal lift $(\tilde{\alpha}, \tilde{\beta})$ of (α, β) is a horizontal curve couple in $T\mathbb{R}^n$ such that $(\tilde{\alpha}, \tilde{\beta})$ is projectable with projection (α, β).

We denote by (α^H, β^H) the horizontal lift of (α, β).

Definition 3: The curves

$$\beta(s) = \alpha(s) + \sum_{i=k+1}^{n} \zeta_i V_i,\ \zeta_i = \zeta_i(s),\ k < n-1 \tag{5}$$

such that the curve α is one of its involutes of order k, are called evolutes of order n–k–1 of the given curve α in $\mathbb{R}^n$.

Definition 4: Given a curve α in $\mathbb{R}^n$, if there exists an evolute β of order n–k–1, then the couple (α, β) is called an involute–evolute curve couple of order (k, n–k–1).

Now we give an application our definitions for the special case $n = 3$. Let $\alpha: I \to \mathbb{R}^3$ be a given curve in $\mathbb{R}^3$. If there exists a curve $\beta: I \to \mathbb{R}^3$ given by

$$\beta(s) = \alpha(s) + \zeta_2 V_2 + \zeta_3 V_3$$

such that the curve α is one of its involutes of order 1, then β is an evolute of order 1 of α. Thus, according to our definitions, (α, β) is an involute–evolute curve couple of order (1, 1) in $\mathbb{R}^3$. Furthermore this couple is unique involute–evolute curve couple in $\mathbb{R}^3$.

Lemma 1: Let X and Y be two vector field on $\mathbb{R}^n$. Then,

$$g^{VH}(X^H, Y^H) = g(X, Y). \tag{6}$$

Proof: From (2) we write

$$g^{VH}(X^H, Y^H) = \sum_{j,i=1}^{n} (g_{ji} + \sum_{h=1}^{n} (\Gamma^h{}_j g^h{}_i + \Gamma^h{}_i\, g_{jh}))\, (X^H)^j (Y^H)^i +$$

$$\sum_{j,i=1}^{n} g_{ji}\, (X^H)^j (Y^H)^{\bar{i}} + \sum_{j,i=1}^{n} g_{ji}\, (X^H)^{\bar{j}} (Y^H)^i.$$

Thus we obtain

$$g^{VH}(X^H, Y^H) = \sum_{j,i=1}^{n} g_{ji}X^j Y^i + \sum_{j,i,h=1}^{n} \Gamma_j^h g_{hi} X^j Y^i + \sum_{j,i,h=1}^{n} \Gamma_i^h g_{jh} X^j Y^i$$

$$- \sum_{j,i,h=1}^{n} \Gamma^i_h g_{ji} X^j Y^h - \sum_{j,i,h=1}^{n} \Gamma^j_h g_{ji} X^h Y^i$$

$$g^{VH}(X^H, Y^H) = \sum_{j,i=1}^{n} g_{ji}X^j Y^i = g(X, Y).$$

Theorem 1: Let (α, β) be an involute–evolute curve couple of order (k, n–k–1) in Riemannian manifold (IR^n, g). Then the horizontal lift curve couple (α^H, β^H) is also an involute–evolute curve couple of same order in pseudo–Riemannian manifold (TIR^n, g^{VH}).

Proof: Let $\{V_1, \ldots, V_n\}$ and $\{W_1, \ldots, W_n\}$ be Frenet n–frames of α and β, respectively. Since (α, β) is an involute–evolute curve couple of order (k, n–k–1) in (IR^n, g), α is an involute of order k of the curve

$$\beta(s) = \alpha(s) + \sum_{i=k+1}^{n} \zeta_i V_i, \; \zeta_i = \zeta_i(s), \; k < n-1,$$

Therefore the curve

$$\alpha(s) = \beta(s) + \sum_{i=1}^{k} \xi_i W_i, \; \xi_i = \xi_i(s), \; k < n-1 \qquad (7)$$

is orthogonal trajectory of the system of the k–dimensional osculating spaces of β, that is, the tangent $V_1 = \dot{\alpha}(s)$ is orthogonal to the osculating spaces Sp $\{W_1, \ldots, W_k\}$ for every s. Thus we have

$$g(V_1, W_r) = 0 \text{ for } 1 \leq r \leq k. \qquad (8)$$

Now we consider the horizontal lift of $\alpha(s)$. From (7) we get

$$\dot{\alpha}^H(s) = \beta^H(s) + \sum_{i=1}^{k} \xi_i^V W_i^H. \qquad (9)$$

Since $\alpha^H(s) = V_1^H$ and $g^{VH}(V_1^H, W_r^H) = g(V_1, W_r)$ from (6), we obtain

$$g^{VH}(V_1^H, W_r^H) = 0 \text{ for } 1 \leq r \leq k. \tag{10}$$

Thus we see that α^H is an orthogonal trajectory of the system of the k-dimensional osculating spaces of β, that is, α^H is in an involute of order k of β^H. Therefore (α^H, β^H) is an involute-evolute curve couple of order (k, n–k–1) in $(T\mathbb{R}^n, g^{VH})$.

Theorem 2: Let (α, β) be an involute–evolute curve couple of order (k, n–k–1) in Riemannian manifold $(\mathbb{R}^n, g)$ with the Riemannian connection ∇. Then.

$$\sum_{i=k}^{n} (\dot{\zeta}_i + \zeta_{i-1} k_{i-1} - \zeta_{i+1} k_i) V_i = \sum_{i=1}^{k+1} (\dot{\xi}_i + \xi_{i-1} \bar{k}_{i-1} - \xi_{i+1} \bar{k}_i) W_i, \tag{11}$$

where k_i and $\bar{k}_i$ are the curvature functions of α and β respectively, such that $\bar{k}_0 = k_0 = \bar{k}_n = k_n = 0$ and the functions ζ_i and ξ_i are defined by (5) and (7) respectively, such that $\zeta i = 0$ for $i < k + 1$ and $\xi_i = 0$ for $i > k$.

Proof: Since (α, β) be an involute–evolute curve couple of order (k, n–k–1) in $(\mathbb{R}^n, g)$, the curve β can be written in the form

$$\beta(s) = \alpha(s) + \sum_{i=k+1}^{n} \zeta_i V_i.$$

Thus we get

$$\dot{\beta}(s) = \dot{\alpha}(s) + \sum_{i=k+1}^{n} \dot{\zeta}_i V_i + \sum_{i=k+1}^{n} \zeta_i \dot{V}_i$$

$$\dot{\beta}(s) = \dot{\alpha}(s) + \sum_{i=k+1}^{n} \dot{\zeta}_i V_i + \sum_{i=k+1}^{n} \zeta_i (-k_{i-1} V_{i-1} + k_i V_{i+1}),$$

where $k_0 = k_n = 0$. In addition, with $\zeta i = 0$ for $i < k + 1$, we obtain

$$\dot{\beta}(s) = \dot{\alpha}(s) + \sum_{i=k}^{n} (\dot{\zeta}_i + \zeta_{i-1} k_{i-1} - \zeta_{i+1} k_i) V_i. \tag{12}$$

Now with the use of the hypothesis for the curve α we can write

$$\alpha(s) = \beta(s) + \sum_{i=1}^{k} \xi_i W_i.$$

Thus we get

$$\dot{\alpha}(s) = \dot{\beta}(s) + \sum_{i=1}^{k} \dot{\xi}_i W_i + \sum_{i=1}^{k} \xi_i \dot{W}_i$$

$$\dot{\alpha}(s) = \dot{\beta}(s) + \sum_{i=1}^{k} \dot{\xi}_i W_i + \sum_{i=1}^{k} \xi_i (-\bar{k}_{i-1} W_{i-1} + \bar{k}_i W_{i+1}),$$

where $\bar{k}_0 = \bar{k}_n = 0$. Furthermore, with $\xi_i = 0$ for $i > k$, we obtain

$$\dot{\alpha}(s) = \dot{\beta}(s) + \sum_{i=1}^{k+1} (\dot{\xi}_i + \xi_{i-1} \bar{k}_{i-1} - \xi_{i+1} \bar{k}_i) W_i \qquad (13)$$

Substituting (12) in (13) we get

$$\sum_{i=k}^{n} (\dot{\zeta}_i + \zeta_{i-1} k_{i-1} - \zeta_{i+1} k_i) V_i = \sum_{i=1}^{k+1} (\dot{\xi}_i + \xi_{i-1} \bar{k}_{i-1} - \xi_{i+1} \bar{k}_i) W_i$$

which completes the proof.

Consider the horizontal lift curve couple (α^H, β^H) of an involute–evolute curve couple (α, β) of order $(k, n-k-1)$ in $(T\mathbb{R}^n, g^{VH})$ with the horizontal lift connection ∇^H. Denote by $\tilde{k}_i$ and $\tilde{\bar{k}}_i$, $\tilde{V}_i$ and $\tilde{W}_i$ the curvature functions of α^H and β^H, the Frenet vector fields of α^H and β^H. Then

$$\tilde{k}_i = k_i^V = k_i o \pi, \ \tilde{\bar{k}}_i = \bar{k}_i^V = \bar{k}_i o \pi, \ \tilde{V}_i = V_i^H, \ \tilde{W}_i = W_i^H$$

(Turgut, 1989).

Thus as in the previous theorem we get

$$\sum_{i=k}^{n} (\dot{\zeta}_i^V + \zeta_{i-1}^V k_{i-1}^V - \zeta_{i+1}^V k_i^V) V_i^H = \sum_{i=1}^{k+1} (\dot{\xi}_i^V + \xi_{i-1}^V \bar{k}_{i-1}^V - \xi_{i+1}^V \bar{k}_i^V) W_i^H. \qquad (14)$$

REFERENCES

GERRETSEN, J.C.H., (1962), Lectures on Tensor Calculus and Differential Geometry, P. Noordhoff N.V. Groningen.

TURGUT, A., (1989), Tanjant Manifold Üzerinde Metrikler, Konneksiyonlar ve Eğriler, (M. Sc. Thesis), Gazi Univ., Institue of Science Technology.

YANO, K., ISHIARA, S., (1973), Tangent and Cotangent Bundles, Marcel Dekker Inc. New York.