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ON THE CLOSED PLANAR HOMOTHETIC MOTIONS AND THE POLAR INERTIA MOMENTUMS OF ORBITS

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ABSTRACT

In this study, the polar inertia momentum of the trajectory curve (X) generated by a fixed point X of the moving plane under the l-parameter closed planar homothetic motion is calculated. And it is seen that all the fixed points of the moving plane the trajectory curves of which have equal polar inertia momentum lie on the same circle of the moving plane such that the center of the circle is generally different from the Steiner point (Theorem 1). Obtaining a a formula which is analogoue of Holditch formula from that we get some corollaries (Corollaries 3, 4 and 5). Moreover we have given a relation between the areas of orbits and the polar inertia momentums of them (Theorem 2).

INTRODUCTION

1. Parameter Closed Planar Homothetic Motions

In this section we are going to consider planes as being complex planes, that is, each point X of the plane is to be observed as the representative of a complex number $x = x_1 + ix_2$. A 1– parameter homothetic motion of plane is described in complex number notation by

$$x' = h\,x\,e^{i\varnothing} + u' \tag{1}$$

where x is a complex number which corresponds to a point given in the so-called moving plane, say E, and x′ is the complex number corresponds to the same point in the so-called fixed plane, say E′, and h = h(t), u′ = u′(t) and ∅ = ∅ (t) are functions of a real single parameter t. Moreover u represents the origin of moving system in the coordinates of the fixed system. Then we have

$$u' = -\,u e^{i\varnothing}. \tag{2}$$

If T is the smallest positive number satisfying the following esualities

$$\begin{aligned} u_j\,(t+T) &= u_j\,(t) \\ \varnothing\,(t+T) &= \varnothing\,(t) + 2\pi\nu \end{aligned} \tag{3}$$

then the motion given by (1) is called 1–parameter closed planar homothetic motion with the period T and rotation number ν[1]. Such a motion will be denoted by B.

For the moving pole points P of the motion B we find that

$$p = p_1 + ip_2 = (\dot{u} + i\ \dot{\varnothing} u) / (\dot{h} + i\ \dot{\varnothing}\ h). \qquad (4)$$

After some calculation, from (1), (3) and (4) we get

$$du_1 = p_1\, dh - p_2 hd\ \varnothing + u_2 d\varnothing,\quad du_2 = p_2\, dh + p_1 hd\ \varnothing - u_1 d\varnothing$$

or (5)

$$u_1 = p_1 h + (p_2 dh) / d\ \varnothing - du_2 / d\varnothing,$$
$$u_2 = p_2\, h - (p_1\, dh) / d\ \varnothing + du_1 / d\ \varnothing.$$

From (1) we can write

$$x'\ \bar{x}' = (h\ x\ e^{i\varnothing} + u')\ (h\bar{x}e^{-i\varnothing} + u')$$

where $\bar{x}'$ is the comlex conjugate of x', that is, $\bar{x}' = x'_1 - ix'_2$. On account of the facts that

$$u' = -ue^{i\varnothing} \text{ and } u' = -ue^{-i\varnothing} \qquad (7)$$

we obtain

$$x'\bar{x}' = u'\bar{u}' - hu\bar{x} - h\bar{u}x + h^2 x\bar{x}. \qquad (8)$$

II. On The Polar Inertia Momentum Of A Curve Drawn By A Fixed Point Of The Moving Plane Under The 1–Parameter Closed Planar Homohetic Motins

Let X be a fixed point of the moving plane and consider the trajetory curve (X) of X under the motion B. The polar inertia momentum of the curve (X), T_X, is given by

$$T_X = \oint x'\ \bar{x}'\ d\varnothing \qquad (9)$$

where the integration is taken along the closed curve (X) in the fixed plane E′ [2]. By using (6), (7) and (8) in (9) we get

$$T_X = T_O + X\bar{X} \oint h^2\ d\varnothing - 2x_1 \oint (p_1 h^2\ d\varnothing + p_2 hdh - hdu_2) -$$
$$- 2x_2 \oint (p_2 h^2\ d\varnothing - p_1\ hdh + hdu_1). \qquad (10)$$

Let us consider the Steiner point $s = s_1 + is_2$ of the moving centrode (p) for the distribution of mass with the density $h^2 d\varnothing$. Hence we have

$$s_j = \frac{\oint p_j h^2 \, d\varnothing}{\oint h^2 \, d\varnothing}, \quad i = 1, 2 \tag{11}$$

where the integrations are taken along the closed curve (P). Let

$$\oint h^2 \, d\varnothing = 2k\Pi. \tag{12}$$

Then, from (10), (11) and (12) we obtain that

$$T_X = T_O + k\Pi\,(x\bar{x} - x\bar{s} - \bar{x}s) - 2x_1 \oint (p_2 hdh - hdu_2) - 2x_2 \oint (hdu_1 - p_1\, hdh). \tag{13}$$

As a result of (13) we can give the following theorem.

Theorem 1: All the fixed points of the moving plane the trajectory curves of which have equal polar inertia momentum lie on the same circle of the moving plane.

On the other hand, for the area F_X enclosed by the trajectory curve (X) we have, [3],

$$F_X = k\Pi\,(x\bar{x} - xs - \bar{x}s) + F_O + \tfrac{1}{2} x_1 \oint (-2p_2 hdh + hdu_2 + u_2 dh \quad \tfrac{1}{2} x_2 \oint (2p_1 hdh - hdu_1 - u_1\, dh) \tag{14}$$

Considering (13) and (14) we find that

$$T_X = T_O + 2\,(F_X - F_O) - x_1 \oint (u_2 dh - hdu_2) - x_2 \oint (hdu_1 - u_1\, dh) \tag{15}$$

Corollary 1. If $h = 1$ then we have

$$-x_1 \oint (u_2\, dh - hdu_2) - x_2 \oint (hdu_1 - u_1\, dh) = 0.$$

In this special case we obtain the relation

$$T_X = 2\,(F_X - F_O) + T_O$$

which is the result given by H.R. Müler [2].

Now, let us consider two fixed points X and Y of the moving plaae, and choose an another fixed point Z on the line segment $\bar{X}Y$, that is

$$z = \lambda x + \mu y, \quad \lambda + \mu = 1$$

Thus, by using (1) it can be easily shown that

$$z' = \lambda x' + \mu y'.$$

For the polar inertia momentum T_Z we have

$$T_Z = \lambda^2\, T_X + \mu^2\, T_Y + 2\lambda\mu T_{XY} \tag{16}$$

where

$$T_{XY} = T_{YX} = 2 \oint (x' \bar{y}' + \bar{x}' y')\, d\varnothing$$

and it is the mixture polar inertia momentum of the trajectory curves (X) and (Y). After some calculation we obtain

$$T_{XY} = T_O + k\Pi\,(xy - xy) - (x_1 + y_1)\, 2k\Pi s_1 - (x_2 + y_2)\, 2k\Pi s_2$$
$$-(x_1 + y_1) \oint (p_2 hdh - hdu_2) - (x_2 + y_2) \oint (-p_1 hdh + hdu_1). \qquad (17)$$

Taking $s = 0$ in (13) and (17) we get

$$T_X = T_S + 2k\Pi xx - 2x_1 \oint (p_2 hdh - hdu_2) - 2x_2 \oint (-p_1\, hdh + hdu_1)$$
$$T_Y = T_S + 2k\Pi yy - 2y_1 \oint (p_2 hdh - hdu_2) - 2y_2 \oint (-p_1\, hdh + hdu_1) \qquad (18)$$
$$T_{XY} = T_S + k\Pi\,(xy + xy) - (x_1 + y_1) \oint (p_2 hdh - hdu_2) -$$
$$(x_2 + y_2) \oint (hdu_1 - p_1 hdh).$$

As a result of (18) we can give the following result:

Corollary 2. If $k > 0$ and $X \neq Y$ then we have the inequality

$$T_X - 2T_{XY} + T_Y > 0\,.$$

Proof: From (18) we obtain

$$T_X - 2T_{XY} + T_Y = 2k\Pi d^2,\ d = |\,XY\,| \qquad (19)$$

where $|\,XY\,|$ is the length of the line segment $\overline{XY}$. So, from (19), the assertion is clear.

Combining (16) and (19) we have

$$T_Z = \lambda T_X + \mu T_Y - 2\lambda\mu k\Pi d^2. \qquad (20)$$

On the other hand the motion does not depend on the choice of the co-ordinate systems. So we can choose the moving system such that the points X and Y lie on the real-axis. Hence, from (20), we obtain

$$T_Z = \frac{1}{d}\,(aT_X + bT_Y) - 2k\Pi ab \qquad (21)$$

where $a = y - z = \lambda d$, $b = z - x = \mu d$, $d = a + b$. If the points X and Y draw the same convex trajectory curve (X) under the motion B then we have $T_X = T_Y$. In this case, from (21), we get

$$T_X - T_Z = 2k\Pi ab. \qquad (22)$$

Thus, we can give the following corollary

Corolary 3. The difference between the polar inertia momentums of the curves (X) and (Z) depends on the homothetic motion because it is a function of k. If $h = 1$ we have $k = \nu$. In this special case, taking $\nu = 1$ we obtain the result given by H.R. Müller [2].

Let us choose another two fixed points Z_1 and Z_2 of the moving plane on the line segment $\overline{XY}$ determined by the two fixed points X and Y of the moving plane. Let two of them, say X and Y, move on the same convex curve (X) while the other two describe different curves (Z_1) and (Z_2) under the motion B. If the difference between the polar inertia momentums of the curves (X) and (Z_1) is T and the difference between the polar inertia momentums of the curves (X) and (Z_2) is T, then, for the ratio T / T', we have

$$\frac{T}{T'} = \left[\frac{|XZ_1|}{|XZ_2|}\right]^2 \cdot \mu \ , \quad \mu = \frac{|XZ_2|}{|XZ_1|} \cdot \frac{|YZ_1|}{|YZ_2|}$$

where the magnitude μ is the cross ratio of the four points X, Y, Z_2 and Z_1, i.e. $\mu = (XYZ_2Z_1)$. Hence we see that the ratio T / T' depends on neither the curve (X) nor the length of the line segment $\overline{XY}$ but only on the relative positions of these four points. So we can give the following corollary

Corollary 4. Let us consider the 1–parameter closed planar homothetic motion and a fixed straight line k on the moving plane. In addition to this, let us choose four points X, Y, Z_1 and Z_2 on the line k such that during the motion two of them move on the same curve (X), while the other two describe different trajectory curves (Z_1) and (Z_2). If the difference between the polar inertia momentums of the curves (X) and (Z_1) is T and the difference between the polar inertia momentums of the curves (X) and (Z_2) is T′ then the ratio T / T' depends only on the relative positions of these four points.

Corollary 5. Let X, Y, A and B be four points of the moving plane such that the polar inertia momentums of the trajectory curves of them are equal and let the line segments XY and AB meet in Z. Then the necessary and sufficient condition for the points X, Y, A and B to lie on the same circle of the moving plane is

$$|XZ| \, . \, |ZY| = |AZ| \, . \, |ZB| \, . \tag{23}$$

Proof: All of the line segment XY, AB, XA, YA, XB, and YB are consant. Using (22) we obtain that

$$T_A - T_Z = 2k\Pi \mid AZ \mid . \mid ZB \mid . \tag{24}$$

Comparison of (22) and (24) yields (23).

Theorem 2. Let X and Y be two choosen fixed points of the moving plane, and Z be an another fixed point on the line segment $\overline{XY}$. If X and Y draw the same closed curve (X) in the fixed plane under the motion, then the ratio of the area of the annulus between the trajectory curves (X) and (Z) to the difference of the polar inertia momentums of the trajectory curves depends on neither the homothetic motion nor the position of Z on the line segment $\overline{XY}$.

Proof: For the area of the ring–shaped region between the trajectory curves (X) and (Z) we have, [3],

$$F_X - F_Z = k\Pi \mid XZ \mid . \mid ZY \mid . \tag{25}$$

From (22) and (25) we obtain that

$$\frac{F_X - F_Z}{T_X - T_Z} = \frac{1}{2}$$

So, the assertion is clear.

ÖZET

1–parametreli kapalı düzlemsel homotetik hareketler için hareketli düzlemde sabit bir noktanın sabit düzlemde çizdiği yörüngenin (eylemsizlik) momenti hesaplandı ve eşit kutupsal atalet momentine sahip yörüngeler çizen bütün sabit noktaların hareketli düzlemde merkezi Steiner noktası olmayan bir çember üzerinde olduğu görüldü (Teorem 1). Atalet momentleri için Holditch formülüne benzer bir bağıntı elde edilerek bazı sonuçlar verildi (Sonuç 3, 4, ve 5). Ayrıca yörünge eğrilerinin alanları ile atalet momentleri arasında bir bağıntı verildi (Teorem 2).

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