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ON THE CURVATURES OF THE PARALLEL HYPERSURFACES

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ABSTRACT

In this paper, we have shown that if (n–2)–th mean curvature M_{n-2} of a hypersurface M is zero, then the sum of principal radii of curvatures of the parallel hypersurface Mr is constant. Secondly, we generalize a theorem of Bonnet which is for the parallel hypersurfaces in E^3, to E^n.

1. INTRODUCTION

In this section, we will give some fundamental definitions and theorems, which are necessary for the following sections.

Definition 1.1: Let M be an oriented hypersurface in E^n. Define a map f as follows:

$$f: M \to E^n$$

$$P \to f(P) = P + rN_P,$$

Where N is the unit normal vector field on M, which gives the orientiation of M, and r is a given real number. Then $M_r = f(M)$ is a hypersurface in E^n and furthermore M_r is called a parallel hypersurface to M, in E^n [4].

Definition 1.2: Let M be a hypersurface in E^n. Let $k_1, \ldots, k_{n-1}$ be the principal curvatures of M. Put

$$\binom{n-1}{s} M_s = \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s}, \; M_0 = 1,$$

where $\binom{n-1}{s} = (n-1)! / (n-1-s)!\, s!$.

We call M_s the s–th mean curvature of M [1]).

Theorem 1.1: Let M and M_r be parallel hypersurfaces in E^n. If k is a principal curvature of M at P, in the direction of X, then $k/(1 + rk)$ is the corresponding principal curvature of M_r at f (P) in the direction of f^* (X), [4].

Theorem 1.2: Let M and M_r be parallel hypersurfaces in E^n. Then

$$H^r = \sum_{i=1}^{n-1} \frac{k_i}{1 + rk_i}$$

and

$$K^r = \prod_{i=1}^{n-1} \frac{k_i}{1 + rk_i}$$

where k_i, $1 \leq i \leq n-1$, denote the principal curvatures of M and H^r and K^r stands for mean and Gaussian curvatures of M_r, respectively [2].

Theorem 1.3: Let M and M_r be parallel surfaces in E^2. If $M \subset E^3$. is a minimal surface (H = 0), then

$$\frac{1}{k_1^r} + \frac{1}{k_2^r} = 2r = \text{constant},$$

where $k^r{}_1$ and $k^r{}_2$ denote principal curvatures of M_r, [3].

The following Theorem due to Bonnet.

Theorem 1.4: (Bonnet): Let M be a surface of constant positive Gauss curvature K with no umbilics. Let $r_1 = \frac{1}{\sqrt{K}}$ and $r_2 = -\frac{1}{\sqrt{K}}$ define parallel sents M_1 and M_2, respectively.

Then,

i) M_1 and M_2 are immersions of M which have constant mean curvatures $\sqrt{K}$ and $-\sqrt{K}$, respectively.

ii) If M is a surface with constant mean curvature H (non zero) and non–zero Gauss curvature, letting $r = -1/H$ yields a parallel set that is an immersion of M with constant positive Gauss curvature H^2, [4].

2. GENERALIZATIONS OF THE THEOREM 1.3 AND THE THEOREM 1.4.

Theorem 2.1: Let M and M_r be parallel hypersurfaces in E^n. If (n–2)–th mean curvature M_{n-2} of M is zero, then

$$\sum_{i=2}^{n-1} \frac{1}{k^r{}_i} = (n-1)\ r = \text{constant},$$

where $k^r{}_i$, $1 \leq i \leq n-1$, denote principal curvature of M_r at the point f (P).

Proof: From the Defition 1.2, (n–2)–th mean curvature M_{n-2} of M is

$$\binom{n-1}{n-2} M_{n-2} = \sum_{1 \leq i_1 < \dots < i_{n-2} \leq n-1} k_{i_1} \dots k_{i_{n-2}}$$

$$= \sum_{i=1}^{n-1} k_1 \dots \hat{k}_i \dots k_{n-1}$$

or

$$(n-1)\ M_{n-2} = \sum_{i=1}^{n-1} k_1 \dots \hat{k}_i \dots k_{n-1},$$

where the symbol $\wedge$ means that the term is omitted. On the otherhand, by the Theorem 1.1. we have

$$k^r{}_i = \frac{k_i}{1 + rk_i}, \quad 1 \leq i \leq n-1 .$$

Now, we can show that,

$$\sum_{i=1}^{n-1} \frac{1}{k^r{}_i} = \frac{\sum_{i=1}^{n-1} k_1 \dots \hat{k}_i \dots k_{n-1} + (n-1)\ r \prod_{i=1}^{n-1} k_i}{\prod_{i=1}^{n-1} k_i} .$$

Since,

$$\sum_{i=1}^{n-1} k_1 \dots \hat{k}_i \dots k_{n-1} = 0$$

thus, we get

$$\sum_{i=1}^{n-1} \frac{1}{k^r_i} = \frac{(n-1)\, r \prod_{i=1}^{n-1} k_i}{\prod_{i=1}^{n-1} k_i}$$

$$= (n-1)\, r = \text{constant},$$

as desired.

Special case, $n = 3$: In this case, we find that,

$\sum_{i=1}^{2} \frac{1}{k^r_i} = 2r =$ constant, which is the same as the Theorem 1.3.

Theorem 2.2: Let M and M_r be parallel hypersurfaces, in E^n. Let M_i, $1 \leq i \leq n-1$, i–th constant mean curvature of M. If the following relation

$$\sum_{i=2}^{n-1} \binom{n-1}{i} (i-1)\, r^i M_i = 1,$$

among the i–th mean curvatures of M holds then, the mean curvature H^r of the hypersurface M_r is equal to constant $(1 / r)$.

Proof: From the Theorem 1.2, we can write

$$H^r = \sum_{i=1}^{n-1} \frac{k_i}{1 + rk_i} \quad .$$

On the otherhand, one can easily show that

$$\sum_{i=1}^{n-1} \frac{k_i}{1 + rk_i} = \frac{1}{r} \left[1 + \frac{-1 + \sum_{s=2}^{n-1} r^s (s-1) \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s}}{1 + \sum_{s=2}^{n-1} r^s \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s}} \right]$$

Since,

$$\binom{n-1}{s} M_s = \sum_{i \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s} \quad .$$

So we have the following,

$$H^r = \frac{1}{r}\left[1 + \frac{-1 + \sum_{s=2}^{n-1} r^s (s-1) \binom{n-1}{s} M_s}{1 + \sum_{s=1}^{n-1} r^s \binom{n-1}{s} M_s}\right]$$

From the hypothesis

$$\sum_{s=2}^{n-1} \binom{n-1}{s} (s-1)\ r^s\ M_s = 1,$$

So we get

$$H^r = \frac{1}{r}$$

which completes the proof.

Special case, $n = 3$: In this case,

$$\sum_{i=2}^{2} \binom{2}{i} (i-1)\ r^i\ M_i = 1$$

or

$$r^2 M_2 = 1$$

From that, we obtain

$$r = \pm \frac{1}{\sqrt{M_2}} .$$

Thus,

$$H^r = \frac{1}{r}$$

$$= \pm \sqrt{M_2}$$

On the otherhand, from the definition of M_2 we know that

$$M_2 = k_1 k_2 = K .$$

So we get

$$H^r = \pm \sqrt{K}$$

that is to say;

$$\text{If } r = \frac{1}{\sqrt{K}}, \text{ then } H^r = \sqrt{K}$$

and

$$\text{If } r = -\frac{1}{\sqrt{K}}, \text{ then } H^r = -\sqrt{K}.$$

This gives us the Theorem 1.4. (i).

Theorem 2.3: Let M and M_r be parallel hypersurfaces in E^n. Denote M_i, $1 \leq i \leq n-1$, for i–th constant mean curvatures of M. If the following relation

$$\sum_{i=1}^{n-2} \binom{n-1}{i} r^i M_i = -1,$$

among the i–th mean curvatures of M holds then, the Gaussian curvature K^r of the hypersurface M_r is equal to $1 / r^{n-2}$.

Proof: From the Theorem 1.2,

$$K^r = \prod_{i=1}^{n-1} \frac{k_i}{1 + rk_i} .$$

On the otherhand, we can calculate that

$$\prod_{i=1}^{n-1} \frac{k_i}{1 + rk_i} = \frac{\prod_{i=1}^{n-1} k_i}{1 + \sum_{s=1}^{n-1} r^s \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s}}$$

Since,

$$\binom{n-1}{s} M_s = \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s},$$

So we can write that,

$$K^r = \frac{M_{n-1}}{1 + \sum_{s=1}^{n-2} \binom{n-1}{s} r^s M_s + r^{n-1} M_{n-1}} .$$

From the hypotehesis,

$$\sum_{i=1}^{n-2} \binom{n-1}{i} r^i M_i = -1 .$$

Thus,

$$K^r = \frac{1}{r^{n-1}} .$$

So we obtain the desired equation.

Special Case, $n = 3$: In this case, from the hypothesis,

$$2rM_1 = -1$$

or

$$r = -\frac{1}{2M_1}$$

On the otherhand, since

$$2M_1 = H$$

we have

$$r = -\frac{1}{H} .$$

Thus, we find the following result,

$$K^r = \frac{1}{r^2}$$

$$= H^2$$

which is the same as Theorem 1.4. (ii).

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