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THE CHARACTERIZATION OF SCHWARZ THEOREM AND UNIT DISCS

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ABSTRACT

Let $\bar{D} = \{z \in C: |z| \leq r\}$ be a set and $A(\bar{D})$ be an algebra of bounded analytic functions on $\bar{D}$. In this paper taking complex algebra R, we gave the characterization of Schwarz theorem. In the special case $r = 1$, we obtained the characterization of Schwarz lemma. Taking $a \in R$ that satisfies some conditions we gave algebraic characterization of conformal mapping from $\bar{D}$ to $\bar{U}$, where $\bar{U} = \{w \in C: |w| \leq 1\}$, and investigate the case $r = 1$.

INTRODUCTION

This paper presents a solution to problem in subject of rings of analytic functions. In late 1940's, it was shown that two domains; D_1 and D_2 in the complex plane, are conformally equivalent iff the rings $B(D_1)$ and $B(D_2)$ of all bounded analytic functions defined on them are algebraically isomorphic. Let R be a ring. It is well know that if R is isomorphic with the ring of bounded analytic functions on an annulus $A = \{z \in C: \rho_1 < |z| < \rho_2\}$, where ρ_1 and ρ_2 are unknown, then it deduces the number ρ_1 / ρ_2 from the ring R [2].

In our study we have taken the known ring and given some algebraic characterizations.

ALGEBRAIC CHARACTERIZATIONS

Let $\varnothing$ be an isomorphism mapping $B(\bar{D})$ onto R. We will denote elements of $B(\bar{D})$ by f, g, f,... and elements of R by a, b, c.... Let e and 1 be multiplicative identy of R and $B(\bar{D})$, respectivelly. Thus, $1 \in B(\bar{D})$ is the function identically equal to 1 on $\bar{D}$. Since $\varnothing : B(\bar{D}) \to R$ is an isomorphism, $\varnothing(1) = e$. Furthermore $\varnothing(n1) = ne$, so that $\varnothing(\pm (m/n).1) = \pm (m/n)\, e$. $-e$ has two square roots in R,one is the image of i. 1, the other is the image of –i. 1. It is algebraically impossible to distinguish between these, since R has an automorphism which takes one into the other (corresponding to the mapping $f \to \overline{f} \in B(\bar{D})$). Thus, we choose one root of –e and make it which correspond to i.1; denote it as ie.

Henceforth, we will denote the complex number field by C and the complex rational number field by C_r. Where a complex number, both real and imaginary parts are real rationals, is called a complex rational number. Clearly, C_r and C are subrings of $B(\bar{D})$.

Lemma 2.1. For each $\alpha \in C_r$, $\varnothing(\alpha) = \alpha$ (or $\bar{\alpha}$).

Proof: If $\alpha \in C_r$, there are the rational numbers r_1 and r_2 such that $\alpha = r_1 + ir_2$. Since $\varnothing(1) = e$ and $\varnothing(i) = i(\text{or } -i)$, we get $\varnothing[(r_1 + ir_2). 1] = r_1e + r_2ie$ (or $r_1e - r_2ie$), ([3], [4]).

Lemma 2.2. For each real number c, $\varnothing(c1) = ce$.

Proof: If c is a rational number, by the Lemma (2.1), $\varnothing(c1) =$ ce. If c is an irrational number, for each rational number c, $c - r \neq 0$.

Thus there exist $(c - r)^{-1} = \dfrac{1}{c-r}$. Then $\varnothing[(c - r). 1] = \varnothing(c1) - re$

and $\varnothing\left[\left(\dfrac{1}{c-r}\right). 1\right] = \dfrac{e}{(\varnothing(c1) - re}$. Therefore $\varnothing(c1) = ce$.

Corollary 2.3. If $c \in C$, $\varnothing(c1) = ce$, [2].

Lemma 2.4. Let $f \in B(\bar{D})$ and let $\bar{R}_f$ be the closed range of f. Then $\lambda \in \bar{R}_f$ iff $f - \lambda 1$ has no inverse in $B(\bar{D})$.

Proof: If $\lambda \in \bar{R}_f$ there is $z_0 \in \bar{D}$ such that $f(z_0) = \lambda$. Then $(f - \lambda 1)(z_0) = 0$. Hence $f - \lambda 1$ has no inverse in $B(\bar{D})$. Now we suppose that $f - \lambda 1$ has no inverse in $B(\bar{D})$. Then at least for one point $z_0 \in \bar{D}$, $(f - \lambda 1)(z_0) = 0$. If follows that $f(z_0) = \lambda$, i.e. $\lambda \in \bar{R}_f$.

Lemma 2.5. $\lambda \in \bar{R}_f$ iff $\varnothing(f) - \lambda e$ has no inverse in R.

Proof: If $\lambda \varnothing \bar{R}_f$, $f - \lambda 1$ has no inverse in $B(\bar{D})$ by Lemma 2.4. Since $\varnothing$ is an isomorphism, $\varnothing(f - \lambda 1) = \varnothing(f) - \lambda e$ has no inverse in R, [1]

Let $\sigma(f)$ and $\sigma(a)$ be spectrum of $f \in B(\bar{D})$ and $a \in R$ respectively. If

$$\rho(a) = \sup \{ |\lambda| : \lambda \in \sigma(a)\},$$

then ρ (a) is also the maximum modulus (Hereinafter abbreviated MM) of $\varnothing^{-1}(a)$.

In this paper, we always consider complex algebra. Now we give first our theorem connected with algebraic characterization.

Theorem 2.6. Let R be a complex algebra, a, b, c $\in$ R and $\varnothing$: $B(\bar{D}) \to R$ be a C–isomorphism. If $\varnothing^{-1}(b) = z$, then $\rho(a) = M$ algebraicly characterizes Schwarz Theorem.

Proof: Let $\varnothing^{-1}(c) = \varphi(z)$, where b, c $\in$ R and a $=$ b.c. Then $\varnothing^{-1}(a) = f(z)$. Since $\varnothing^{-1}(a) = \varnothing^{-1}(b)\ \varnothing^{-1}(c)$, we obtain $f(z) = z.\varphi(z)$. We can write from here

$$\varphi(z) = \frac{f(z)}{z} ,$$

for $z \neq 0$.

For $\varphi(z)$ to be in $B(\bar{D})$, f(z) must be zero at $z = 0$, i.e $f(0) = 0$. Because, as $f(0) = 0$ the point $z = 0$ is a removable singular point for the function $\varphi(z)$. Hence, for each z, $\varphi(z) \in B(\bar{D})$. By the maximum modulus principle in a disk that concentric with $\bar{D}$ and has a radii $k < r$,

$$| \varphi(z) | \leq \frac{M}{k} ,$$

because $\rho(a) = MM(\varnothing^{-1}(a)) = M$. It follows from that for $k \to r$

$$| \varphi(z) | \leq \frac{M}{r}$$

that is,

$$| f(z) | \leq \frac{M}{r} | z | .$$

If we take $M = 1$ and $r = 1$ as a result of Theorem 2.6, we obtain an algebraic characterization of Schwarz Lemma. More clearly,

Corollary 2.7. Let R be complex algebra a, b, c $\in$ R and $\varnothing$: $B(\bar{U}) \to R$ be C–isomorphism. If $\varnothing^{-1}(b) = z$, then $\rho(a) = 1$ algebraicly characterizes Schwarz Lemma.

Another result of Theorem 2.6 is the following.

Corollary 2.8. Let $B(\bar{D})$ be a complex algebra of the bounded analytic functions on $\bar{D}$ and $f \in B(\bar{D})$ be schlicht. Furthermore suppose that $f(0) = 0$ and $MM(f) = 1$. Then,

$$f(z) = \frac{1}{r} \exp i\theta . z$$

where $\bar{D} = \{z \in C: |z| \leq r\}$.

Proof: Since $w = f(z)$ schlicht, $z = f^{-1}(w) \in B(\bar{U})$. Then, we deduce

$$|f(z)| \leq \frac{M}{r} |z|$$

by the Schwarz Theorem.

Since f is the function from $\bar{D}$ to $\bar{U}$, we obtain

$$|f(z)| \leq \frac{1}{r} |z|$$

for $M = 1$ and hence $r|w| \leq |z|$.

Conversely, since the mapping $z = f^{-1}(w)$ maps the closed unity ball to $\bar{D}$, $M = r$ and $r = 1$. Thus,

$$|f^{-1}(w)| \leq \frac{r}{1} . |w|$$

and from here we get $|z| \leq r|w|$. We find $r|w| = |z|$ from both inequalities or

$$\left|\frac{w}{z}\right| = \frac{1}{r} .$$

If follows for that

$$f(z) = \frac{1}{r} \exp i\theta . z$$

The mapping $f(z) = \frac{1}{r} \exp i\theta . z$ maps $\bar{D}$ to $\bar{U}$ such that $f(0) = 0$.

Now we will give an algebraic characterization of f which maps conformally $\bar{D}$ onto $\bar{U}$ such that $f(\alpha) = 0$, where α is interior point of $\bar{D}$.

We need the following Lemma.

Lemma 2.9. Let $\alpha \in \bar{D}$ be. Suppose that $f \in B(\bar{D})$ satisfies the following conditions.

a) $f(\alpha) = 0$,

b) $MM(f) = 1$,

c) f is schlicht.

Then,

$$f(z) = \lambda . \frac{z - \alpha}{r^2 - \bar{\alpha} z}, \tag{2.9.1}$$

where $|\lambda| = r$ and $\bar{D} = \{z \in C : |z| \leq r\}$.

Proof: $I_\alpha = \{f \in B(\bar{D}) : f(\alpha) = 0\}$ is the maximal ideal of $B(\bar{D})$. I_α is generated by $h(z) = z - \alpha$, i.e, $I_\alpha = < z - \alpha >$. The function that we are looking for must be in I_α. If $\alpha = 0$, by Corollary 2.8 $f(z) = \lambda . \frac{z}{r^2}$. If $\alpha \neq 0$, for any z in $\bar{D}$ $MM(z - \alpha) \neq 1$. Therefore $f(z) \neq z - \alpha$. If $f(z) = (z - \alpha)\ g\ (z)$, $f(\alpha) = 0$ and $MM(f) = 1$, then $g(z)$ must be $\frac{\lambda}{r^2 - \bar{\alpha} z}$, where $r = |\lambda|$. Thus

$$f(z) = \lambda . \frac{z - \alpha}{r^2 - \bar{\alpha} z},$$

where $r = |\lambda|$.

Furthermore if f is schlicht, $f(\alpha) = 0$ and $MM(f) = 1$, then this function must be in the form of (2.9.1), [5].

Theorem 2.10. Let R be any algebra such that $\varnothing$ is an isomorphism from $B(\bar{D})$ to R. Furthermore, suppose that the following conditions are satisfied for some $a \in R$.

a) For each $\lambda \in \sigma(a) = \bar{U}$, there is only one point z_0.

b) For each $\alpha \in C$, $< b - \alpha e >$ is a maximal ideal of R. Furthermore, $\varnothing^{-1}(b) = z$ and $a \in < b - \alpha e >$, where $b \in R$.

c) $\rho(a) = MM(\varnothing^{-1}(a)) = 1$.

Then $\varnothing^{-1}(a)$ is a conformally mapping from $\bar{D}$ to $\bar{U}$ and

$$\varnothing^{-1}(a) = \lambda \frac{z - \alpha}{r^2 - \bar{\alpha} z},$$

where $|\lambda| = r$.

Proof: Since $a \in < b - \alpha e >$, there is an element $c \in R$ such that $(b - \alpha e)\, c = a$. Since $\varnothing$ is isomorphism, we can write $\varnothing^{-1}(a) = \varnothing^{-1}(b - \alpha e)$. $\varnothing^{-1}(c)$ and $\varnothing^{-1}(a) = \{\varnothing^{-1}(b) - \varnothing^{-1}(\alpha e)\}\ \varnothing^{-1}(c)$. Thus we find

$$\varnothing^{-1}(a) = (z - \alpha)\ \varnothing^{-1}(c).$$

By the Lemma 2.9, MM $(\varnothing^{-1}(a)) = 1$ and hence

$$\varnothing^{-1}(c) = \frac{\lambda}{r^2 - \bar{\alpha} z}.$$

Clearly, $\varnothing^{-1}(c) \in B(\bar{D})$. We obtain

$$c = \frac{\varnothing(\lambda)}{\varnothing(r^2) - \varnothing(\bar{\alpha} z)} = \frac{\lambda e}{rere - \bar{\alpha} ebe}$$

from the equality and so $c \in R$. Thus

$$a = (b - \alpha e).\ \frac{\lambda e}{r^2 e - \bar{\alpha} ebe} \in (b - \alpha e)$$

and we deduce the mapping

$$\varnothing^{-1}(a) = \lambda\ \frac{z - \alpha}{r^2 - \bar{\alpha} z}.$$

It is well know that this is the mapping from $\bar{D}$ onto $\bar{U}$. At the same time, the mapping $\varnothing^{-1}(a)$ is unique. Because, $\lambda_0 \in \bar{R}_{\varnothing^{-1}(a)}$, by $\lambda_0 \in \sigma(a)$. Since each a point $\bar{R}_{\varnothing^{-1}(a)}$ correspond to unique z_i by the Lemma 2.4 and (a), $\varnothing^{-1}(a) \in B(\bar{D})$ is one–to–one. Since $\varnothing$ is an isomorphism and $< b - \alpha e >$ is maximal principal ideal in R, $\varnothing^{-1}(b - \alpha e)$ is a maximal principal ideal in $B(\bar{D})$. This maximal principal ideal is generated by the $\varnothing^{-1}(b) - \varnothing^{-1}(\alpha e) = z - \alpha$. Then $\varnothing^{-1}(a) \in < z - \alpha >$ by (b). $\varnothing^{-1}(a)$ is schlicht. Thus

$$\varnothing^{-1}(a) = \lambda\ \frac{z - \alpha}{r^2 - \bar{\alpha} z},$$

by Lemma 2.9.

Corollary 2.11. Let R be any algebra and $\varnothing : B(\bar{U}) \to R$ be a C–isomorphism. Furthermore suppose that the following conditions hold.

a) For each $\lambda_0 \in \sigma(a) = \bar{U}$, there is an unique $z_0 \in \bar{U}$.

b) For ecah $\alpha \in C$, $< b - \alpha e >$ is maximal ideal of R, where $b \in R$, $\varnothing^{-1}(b) = z$ and $a \in < b - \alpha e >$.

c) $\rho(a) = MM(\varnothing^{-1}(a)) = 1.$

Then $\varnothing^{-1}(a)$ is conformally mapping from $\tilde{U}$ onto U and

$$\varnothing^{-1}(a) = \lambda \frac{z - \alpha}{1 - \bar{\alpha} z}, \quad (|\lambda| = 1).$$

Proof: This corollary is the special case of Theorem 2.10 for $r = 1$.

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