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ON THE PAIR OF AXOIDS UNDER THE SYMMETRIC HELICAL MOTION OF ORDER k IN THE EUCLIDEAN SPACE E^n

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ABSTRACT

The purpose of this paper, after giving a summary of known results about helical motion of order k and axoids in Euclidean space E^n, is to define the symmetric helical motion of order k and to obtain some results about integral invariants of the pair of axoids under the motion.

1. HELICAL MOTION OF ORDER k

A one parameter motion of a body in Euclidean space E^n is generated by the transformation

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix}, \quad AA^T = I_n \tag{1.1}$$

where $A: J \to SO(n)$ and $C: J \to IR^n$ are functions of differentiability class C^r $(r \geq 3)$ on real interval J. $\bar{x}$ and x correspond to the position vectors of the same point with respect to orthonormal coordinate systems of the moving space $\bar{E}$ and fixed space E, respectively, [4].

The equation (1.1) by differentiation with respect to $t \in J$ yields

$$\dot{x} = B(x-C) + \dot{C}, \quad B = \dot{A}A^{-1}, \quad \dot{\bar{x}} = 0. \tag{1.2}$$

Since the matrix A is orthogonal the matrix B is skew. Therefore only in the case of even dimension it is possible that the determinant $|B|$ may not vanish. If $|B| \neq 0$ in $t \in J$, we get exatly one solution Q(t) of the equation

$$B(Q-C) + \dot{C} = 0. \tag{1.3}$$

In this case, Q is the center of the instantaneous rotation of the motion and called the pole of the motion in t. At the pole Q, the velocity

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vector vanishes by the equation (1.2). Therefore we get a differentiable curve $\alpha:J \to E$ of poles in the fixed space E, called the **fixed pole** curve. By (1.1) there is uniquely determined the **moving pole** $\bar{\alpha}:J \to \bar{E}$ from the fixed pole curve, point to point. If $|B| = 0$, we obtain by the rules of Linear Algebra:

For every $t \in J$ there exist a unit vector $e(t) \in \text{kern}B$ and $\lambda(t) \in IR$ so that the solutions Q of the equation

$$B(Q-C) + \dot{C} = \lambda e \tag{1.4}$$

fill a uniquely determined linear subspace $E_k(t) \subset E^n$ with the dimension $k = n - \text{rank}B$. $E_k(t)$ is the axis of the instantaneous screw $(\lambda \neq 0)$ of the motion or the axis of the insantaneous rotation $(\lambda = 0)$ and will be called **the instantaneous axis** of the motion in $t \in J$, [3]. In this second case, we obtain a generalized ruled surface of dimension $k + 1$ in E generated by the instantaneous axis $E_k(t)$, $t \in J$, which we call the **fixed axoid** $\varnothing$ of the mation. The fixed axoid $\varnothing$ determines the **moving axoid** $\overline{\varnothing}$ in $\bar{E}$ generator to generator by (1.1). The axoids $\varnothing$ ve $\overline{\varnothing}$ of a motion in E^n touch each other along every common pair $E_k(t) \subset \varnothing$ and $\bar{E}_k(t) \subset \overline{\varnothing}$ for all $t \in J$ by rolling and sliding upon each other under the motion, [5]. Such motion is called an **(instantaneously) helical motion of order k** in E^n, [5].

2. GENERALIZED RULED SURFACES

In any k-dimensional generator $E_k(t)$ of a $(k+1)$ dimensional generalized ruled surface (axiod, in [2] "(k + 1)-Regelflache") $\varnothing \subset E^n$ there exist a maximal linear subspace $K_{k-m}(t) \subset E_k(t)$ of dimension k-m with the property that in every point of $K_{k-m}(t)$ no tangent space of $\varnothing$ is determined ($K_{k-m}(t)$ contains all singularities of $\varnothing$ in $E_k(t)$) or there exists a maximal linear subspace $Z_{k-m}(t) \subset E_k(t)$ of dimension k-m with the property that in every point of Z_{k-m} the tangent space of $\varnothing$ is orthogonal to the asymptotic bundle of the tangent spaces in the points of infinity of $E_k(t)$ (all points of $Z_{k-m}(t)$ have the same tangent space of $\varnothing$). We call $K_{k-m}(t)$ the **edge space** in $E_k(t) \subset \varnothing$ and $Z_{k-m}(t)$ the **central space** in $E_k(t) \subset \varnothing$. A point of $Z_{k-m}(t)$ is called a **central point.** If $\varnothing$ possesses generators all of the same type the edge spaces resp. the central spaces generate a generalized ruled surface contained in $\varnothing$ which call the **edge ruled surface** resp. the **central ruled surface.** For $m = k$ the edge ruled surface degenerates in the edge of $\varnothing$, the central ruled surface in the **line of striction.** So the ruled surface with edge ruled generalize the tangent surfaces of E^3, the ruled surface with central ruled surface generalize the ruled surfaces with line of striction of E^3.

For the analytical represention of a $(k+1)$–dimensional ruled surface $\varnothing$ we choose a leading curve α in the edge resp. central ruled surface $\Omega \subset \varnothing$ transversal to the generators. In [2] it is shown that there exists a distinguished moving orthonormal frame (ONF) of $\varnothing$ $\{e_1, \ldots, e_k\}$ with the properties:

(i) $\{e_1, \ldots, e_k\}$ is an ONF of the $E_k(t) \subset \varnothing$,

(ii) $\{e_{m+1}, \ldots, e_k\}$ is an ONF of the $K_{k-m}(t)$ rasp. $Z_{k-m}(t) \subset E_k(t)$,

(iii) $\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij}\, e_j + K_i\, a_{k+i}\,,\ 1 \leq i \leq m,$

$$\dot{e}_{m+p} = \sum_{l=1}^{m} \alpha_{(m+k)\,l}\, e_l\,,\ \text{with } K_i > 0,\ \alpha_{ij} = -\alpha_{ji}, \tag{2.1}$$

$\alpha_{(m+k)\,(m+x)} = 0,\ 1 \leq p, x \leq k\text{–}m,$

(iv) $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}\}$ is an ONF.

A moving ONF of $\varnothing$ with the properties (i)–(iv) is called a **principal frame** of $\varnothing$. If $K_1 > \ldots > K_k > 0$, the principal frame of $\varnothing$ is determined up to the signs. By a given principal frame the vectors $a_{k+1}, \ldots, a_{k+m}$ are well defined.

A leading curve α of $(k+1)$–dimensional ruled surface $\varnothing$ is a leading curve of the edge resp. central surface $\Omega \subset \varnothing$ too iff its tangent vector has the form

$$\dot{\alpha}(t) = \sum_{i=1}^{k} \zeta_i e_i + \eta_{m+1}\, a_{k+m+1}\,, \tag{2.2}$$

where $\eta_{m+1} \neq 0$, a_{k+m+1} is a unit vector well defined up to the sign with the property that $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\}$ is an ONF of the tangential bundle of $\varnothing$. One shows: $\eta_{m+1}(t) = 0$, in $t \in J$ iff the generator $E_k(t) \subset \varnothing$ contains the edge space $K_{k-m}(t)$. If $\eta_{m+1}(t) \neq 0$, we call the m–magnitudes

$$P_i = \frac{\eta_{m+1}}{K_i}\,,\ 1 \leq i \leq m \tag{2.3}$$

the **principal parameters of distribution**. These parameters are direct generalizations of the parameter of distribution of the ruled surface in E^3 (see [2]). A $(k+1)$–dimensional ruled surface with central ruled surface and no principal parameter of distribution $(m = 0)$ is a $(k+1)$–dimensional cylinder.

Moreover the parameter of distribution of a generalized ruled surface $\varnothing$ given in [3] by

$$P = \sqrt[m]{|P_1 P_2 \ldots P_m|} \tag{2.4}$$

and the total parameter of distribution of $\varnothing$ can be dedined in [5] by

$$D = \prod_{i=1}^{m} P_i. \tag{2.5}$$

Suppose that $\varnothing_i$, $1 \leq i \leq k$, are 2–dimensional closed principal ruled surfaces such that the generators of $\varnothing_i$ have the direction of the unit vectors $e_i(t)$, $1 \leq i \leq k$. Then, in the case $m = k$, there exist **k–pitches** given by

$$L_i = -\int_0^p \zeta_i(t)dt, \ 1 \leq i \leq k, \tag{2.6}$$

where $p \in IN$ denotes a period of the motion.

Let $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\}$ be ONF of the tangential bundle $T(t)$ of $\varnothing$. If we complete this ONF by an arbitrary $\{a_{k+m+2}, \ldots, a_n\}$ of the orthogonal complement, called a **complementary** ONF. From the orthogonality conditions, then we obtain by differentiation, [3]:

$$\dot{a}_{k+i} = -K_i e_i + \sum_{j=1}^{m} \tau_{ij} a_{k+j} + w_i a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{i\lambda} a_{k+m+\lambda}, \ 1 \leq i \leq m. \tag{2.7}$$

Suppose that $\dim T(t) = k + m + 1$. If $\varnothing$ is a closed ruled surface, the **m–apex angles** of $\varnothing$ can be define by

$$\lambda_i = \int_0^p w_i(t)\, dt, \ 1 \leq i \leq m, \tag{2.8}$$

and also the apex angle of $\varnothing$ is defined, in [6], by

$$\lambda = \sqrt[m]{|\lambda_1 \lambda_2 \ldots \lambda_m|}. \tag{2.9}$$

3. THE PAIR OF AXOIDS UNDER THE SYMMETRIC HELICAL MOTION

Let $\bar{\alpha} \subset \bar{E}$ and $\alpha \subset E$ be moving and fixed pole curves of the helical motion of order k. Suppose that $\{\bar{e}_1(t),..,\bar{e}_k(t)\}$ is an ONF system at $\bar{\alpha}(t)$ and let $\bar{E}_k(t) = Sp\ \{\bar{e}_1(t),\ldots,\ \bar{e}_k(t)\}$. Then $\bar{E}_k(t)$ generates the moving axoid $\bar{\varnothing}$ with the leading curve $\bar{\alpha}$ in $\bar{E}$.A parametrization of $\bar{\varnothing}$ is

$$\bar{\varnothing}\ (t, \bar{u}_1,\ldots,\bar{u}_k\} = \bar{\alpha}(t) + \sum_{i=1}^{k} \bar{u}_i\bar{e}_i(t),\ \bar{u}_i \in IR,\ t \in J. \tag{3.1}$$

Let $\{e_1(t),\ldots,\ e_k(t)\}$ be an ONF system satisfaying the following equation at the point $\alpha(t)$ in the fixed space E:

$$A\bar{e}_i = -e_i,\ 1 \leq i \leq k. \tag{3.2}$$

$E_k(t) = Sp\ \{e_1(t),\ldots,\ e_k(t)\}$ generates the fixed axoid $\varnothing$ with leading curve α in E by (1.1). And also a parametrization of $\varnothing$ is.

$$\varnothing\ (t, u_1,\ldots,\ u_k) = \alpha(t) + \sum_{i=1}^{k} u_ie_i(t),\ u_i \in IR,\ t \in J. \tag{3.3}$$

Definiton 3.1. If a helical motion given by (1.1) satisfies the equation (3.2), then the motion is called a **symmetric helical motion of order k.**

Let $\bar{\varnothing}$ and $\varnothing$ be (k + 1)– dimensional moving and fixed axoids with the leading curves $\bar{\alpha}$ and α, resp. ($\bar{\alpha}$ and α are the pole curves of the motion). Then we have the following equations, [1]:

$$\dot{\alpha} = A\dot{\bar{\alpha}}, \tag{3.4}$$

$$s = \bar{s}, \tag{3.5}$$

where $\bar{s}$ and s lengthes of $\bar{\alpha}$ and α, respectively.
Then we have the following theorem.

Theorem 3.2. Under the symmetric helical motion of order k the moving and fixed axoids touch each other along every common pair $\bar{\alpha}$ and α for all $t \in J$ by rolling and sliding upon each other.

Let $\bar{E}_k(t)$ and $E_k(t)$ be the generator spaces of the axoids $\bar{\varnothing}$ and $\varnothing$, respectively. From (3.2) we have

$$\dot{A}\bar{e}_i + A\dot{\bar{e}}_i = -\bar{e}_i,\ 1 \leq i \leq k,$$

$$-Be_i + A\dot{\bar{e}}_i = -\dot{e}_i,$$

$$A\dot{\bar{e}}_i = -\dot{e}_i,\ 1 \leq i \leq k,\ (Be_i = 0). \tag{3.6}$$

Then we immediately read off from (3.2) and (3.6).

Theorem 3.3. Under the symmetric helical motion of order k, the generator spaces $\bar{E}_k(t)$ and $E_k(t)$ correspond to each other by the equations (3.2) and (3.6).

Let $\bar{A}(t)$ and $A(t)$ be the asymptotic bundles, with respect to the $\bar{E}_k(t)$ and $E_k(t)$, of the axoids $\bar{\varnothing}$ and $\varnothing$ resp. Then $\bar{A}(t)$ and $A(t)$ can be given resp. by

$$\bar{A}(t) = Sp\ \{\bar{e}_1, \ldots, \bar{e}_k, \dot{\bar{e}}_1, \ldots, \dot{\bar{e}}_k\}, \tag{3.7}$$

$$A(t) = Sp\ \{e_1, \ldots, e_k, \dot{e}_1, \ldots, \dot{e}_k\}. \tag{3.8}$$

Suppose that dim $\bar{A}(t)$ $(= \dim A(t)) = k + m$, $0 \leq m \leq k$, then m vectors of $\dot{\bar{e}}_1, \dot{\bar{e}}_2, \ldots, \dot{\bar{e}}_k$ are linearly independent. Let the linearly independent vectors are renumbered as $\dot{\bar{e}}_{k+1}, \dot{\bar{e}}_{k+2}, \ldots, \dot{\bar{e}}_{k+m}$. Then the set

$$\{\bar{e}_1, \ldots, \bar{e}_k, \dot{\bar{e}}_{k+1}, \ldots, \dot{\bar{e}}_{k+m}\} \tag{3.9}$$

is a basis of the asymptotic bundle $\bar{A}(t)$. Similarly, we get a basis for the asymptotic bundle $A(t)$ as follows

$$\{e_1, \ldots, e_k, \dot{e}_{k+1}, \ldots, \dot{e}_{k+m}\}. \tag{3.10}$$

By the Gram–Schmidt process form (3.9) and (3.10) we get the following orthogonal bases for $\bar{A}(t)$ and $A(t)$ resp.,

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{y}_{k+1}, \ldots, \bar{y}_{k+m}\}, \tag{3.11}$$

$$\{e_1, \ldots, e_k, y_{k+1}, \ldots, y_{k+m}\}. \tag{3.12}$$

Under the symmetric helical motion of order k, the above orthogonal systems correspond to each other by the equation

$$A\bar{y}_{k+j} = -y_{k+j},\ 1 \leq j \leq m. \tag{3.13}$$

If we set

$$\bar{a}_{k+j} = \frac{\bar{y}_{k+j}}{\|\bar{y}_{k+j}\|},\ a_{k+j} = \frac{y_{k+j}}{\|y_{k+j}\|},\ 1 \leq j \leq m,$$

then we get the following ONFs for $\bar{A}(t)$ and $A(t)$resp.,

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}\}, \tag{3.15}$$

$$\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}\}. \tag{3.16}$$

Therefore we have the following theorem.

Theorem 3.4. Under the symmetric helical motion of order k, the asymptotic bundles $\bar{A}(t)$ and $A(t)$ correspond to each other by the following equations:

$$A\bar{e}_i = -e_i,\ 1 \leq i \leq k,$$

$$A\bar{a}_{k+j} = -a_{k+j},\ 1 \leq j \leq m. \tag{3.17}$$

Let $\overline{T}(t)$ and $T(t)$ be the tangential bundles of $\bar{\varnothing}$ and $\varnothing$ resp. If $\dim\overline{T}(t)$ $(= \dim T(t)) = k + m + 1$, then

$$\{\bar{e}_1, \ldots, \bar{e}_k, \dot{\bar{e}}_{k+1}, \ldots, \dot{\bar{e}}_{k+m}, \dot{\bar{\alpha}}\} \tag{3.18}$$

is a basis $\overline{T}(t)$ and

$$\{e_1, \ldots, e_k, \dot{e}_{k+1}, \ldots, \dot{e}_{k+m}, \dot{\alpha}\} \tag{3.19}$$

is a basis for $T(t)$. Using the Gram–Schmidt process, we get following ONFs for $\overline{T}(t)$ and $T(t)$ resp.

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}, \bar{a}_{k+m+1}\}, \tag{3.20}$$

$$\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\}. \tag{3.21}$$

We can give the following theorem.

Theorem 3.5. Under the symmetric helical motion of order k, the tangential bundles $\overline{T}(t)$ and $T(t)$ correspond to each other by the following equations:

$$A\bar{e}_i = -e_i,\ 1 \leq i \leq k,$$

$$A\bar{a}_{k+j} = -a_{k+j},\ 1 \leq j \leq m,$$

$$A\bar{a}_{k+m+1} = a_{k+m+1}. \tag{3.22}$$

Now we can complete the ONF $\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}, \bar{a}_{k+m+1}\}$ of $\overline{T}(t)$ to the ONF

$$\{\bar{e}_1, \ldots, \bar{e}_k, \bar{a}_{k+1}, \ldots, \bar{a}_{k+m}, \bar{a}_{k+m+1}, \ldots, \bar{a}_n\} \tag{3.23}$$

of E^n. The orthonormal complement

$$\{\bar{a}_{k+m+2}, \ldots, \bar{a}_n\} \tag{3.24}$$

is called a complementary ONF of $\overline{\varnothing}$.
If we set

$$A\bar{a}_{k+m+\lambda} = y_{k+m+\lambda},\ 2 \leq \lambda \leq n-k-m, \tag{3.25}$$

then we get an orthogonal complement $\{y_{k+m+2},\ldots, y_n\}$ of $\varnothing$ under the symmetric helical motion of order k. If we set

$$a_{k+m+\lambda} = \frac{y_{k+m+\lambda}}{\|y_{k+m+\lambda}\|},\ 2 \leq \lambda \leq n-k-m, \tag{3.26}$$

then we have the following orthonormal complementary ONF of $\varnothing$

$$\{a_{k+m+2},\ldots, a_n\}. \tag{3.27}$$

Theorem 3.6. Under the symmetric helical motion of order k, the complementary ONFs (3.24) and (3.27) satisfy the following equation:

$$A\bar{a}_{k+m+\lambda} = a_{k+m+\lambda},\ 2 \leq \lambda \leq n-k-m.$$

Therefore, for the symmetric helical motion of order k, we can give the following two corollaries:

Corollary 3.7. $\overline{T}(t)$ and $T(t)$ being two tangential bundles which are correspond to each other under the symmetric helical motion of order k. Let $\{\bar{e}_1,\ldots,\bar{e}_k,\ \bar{a}_{k+m},\ \bar{a}_{k+m+1},\ldots, \bar{a}_n\}$ and $\{e_1,\ldots, e_k,\ a_{k+m},\ a_{k+m+1},\ldots, a_n\}$ be two ONFs of E^n with respect to the $\overline{T}(t)$ and $T(t)$ resp. Then we have the following equations: (3.2), (3,17), and

$$A\bar{a}_{k+m+\lambda} = a_{k+m+\lambda},\ 1 \leq \lambda \leq n-k-m. \tag{3.28}$$

Corollary 3.8. A symmetric helical motion of order k of E^n is a reflection with respect to the subspace $Sp\{\bar{a}_{k+m+1},\ldots,\bar{a}_n\}$ of dimension (n–k–m).

4. THE INTEGRAL INVARIANTS OF THE PAIR OF AXOIDS WHICH CORRESPOND TO EACH OTHER UNDER THE SYMMETRIC HELICAL MOTION OF ORDER k

Theorem 4.1. Let $\overline{\varnothing}$ and $\varnothing$ be the $(k+1)$– dimensional moving and fixed axoids which correspond to each other under the symmetric helical motion with the leading curves $\bar{\alpha}$ and α resp., $\{\bar{e}_1,\ldots, \bar{e}_k\}$ and $\{e_1,\ldots, e_k\}$ being the principal ONFs of $\overline{\varnothing}$ and $\varnothing$ resp., we have

$$\bar{\zeta}_i = -\zeta_i,\ 1 \leq i \leq k, \tag{4.1}$$

$$\overline{\eta}_{m+1} = \eta_{m+1}, \tag{4.2}$$

where $\dot{\bar{\alpha}} = \sum_{i=1}^{k} \overline{\zeta}_i \bar{e}_i + \overline{\eta}_{m+1} \bar{a}_{k+m1}$ and $\dot{\alpha} = \sum_{i=1}^{k} \zeta_i e_i + \eta_{m+1} a_{k+m+1}$.

Proof:

$$A(\dot{\bar{\alpha}}) = A\left(\sum_{i=1}^{k} \overline{\zeta}_i \bar{e}_i + \overline{\eta}_{m+1} \bar{a}_{k+m+1} \right),$$

$$A(\dot{\bar{\alpha}}) = \sum_{i=1}^{k} \overline{\zeta}_i A(\bar{e}_i) + \overline{\eta}_{m+1} A(\bar{a}_{k+m+1}). \tag{4.3}$$

Using (3.2), (3.22), and (3.4) the theorem is proved.

Theorem 4.2. For

$$\dot{\bar{a}}_{k+i} = -\overline{K}_i \bar{e}_i + \sum_{i=1}^{m} \overline{\tau}_{ij} \bar{a}_{k+j} + \overline{w}_i \bar{a}_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \overline{\gamma}_{i\lambda} \bar{a}_{k+m+\lambda}, \; 1 \leq \imath \leq m, \tag{4.4}$$

$$\dot{a}_{k+i} = -K_i e_i + \sum_{i=1}^{m} \tau_{ij} a_{k+j} + w_i a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{i\lambda} a_{k+m+\lambda}, \; 1 \leq i \leq m, \tag{4.4}$$

we have

$$\overline{K}_i = K_i, \; \overline{w}_i = -w_i, \; \overline{\gamma}_{i\lambda} = -\gamma_{i\lambda}, \; 1 \leq i \leq m, \; 2 \leq \lambda \leq n\text{–}k\text{–}m. \tag{4.5}$$

Proof:

$$A(\dot{\bar{a}}_{k+i}) = A\left[-\overline{K}_i \bar{e}_i + \sum_{j=1}^{m} \overline{\tau}_{ij} \bar{a}_{k+j} + \overline{w}_i \bar{a}_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \overline{\gamma}_{i\lambda} \bar{a}_{k+m+\lambda} \right].$$

Since A linear, using (3.2), (3.17), (3.22), and (3.28) we get.

$$A\dot{\bar{a}}_{k+i} = \overline{K}_i e_i - \sum_{j=1}^{m} \overline{\tau}_{ij} a_{k+j} + \overline{w}_i a_{k+m+1} + \sum_{\lambda=2}^{u-k-m} \overline{\gamma}_{i\lambda} a_{k+m+\lambda}. \tag{4.6}$$

From (3.17)

$$\dot{A}\bar{a}_{k+i} + A\dot{\bar{a}}_{k+i} = -\dot{a}_{k+i},$$

$$A\dot{\bar{a}}_{k+i} = -\dot{a}_{k+i} - \dot{A}\bar{a}_{k+i},$$

$$A\dot{\bar{a}}_{k+i} = -\dot{a}_{k+i} + \dot{A}A^{-1}a_{k+i} \; (\bar{a}_{k+i} = -A^{-1}a_{k+i}),$$

$$\dot{a}_{k+i} = -A\dot{\bar{a}}_{k+i} + Ba_{k+i} \; (\dot{A}A^{-1} = B), \; 1 \leq i \leq m. \tag{4.7}$$

If we set (4.7) in (4,4)', then we obtain

$$A\dot{\bar{a}}_{k+i} = K_i e_i - \sum_{j=1}^{m} \tau_{ij} a_{k+j} - w_i a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{i\lambda} a_{k+m+\lambda} + Ba_{k+i}. \tag{4.8}$$

Therefore from (4.6) and (4.8), the teorem is proved.

Theorem 4.3. If $\bar{P}_i$ and P_i principal parameters of distribution of the axoids $\bar{\varnothing}$ and $\varnothing$ resp., then

$$\bar{P}_i = P_i, \; 1 \leq i \leq m. \tag{4.9}$$

Proof: Using (4.2) and (4.5) in $\bar{P}_i = \bar{\eta}_{m+1} / \bar{K}_i$, the theorem is proved.

Corollary 4.4. For the axoids $\bar{\varnothing}$ and $\varnothing$,

$$\bar{P} = P, \tag{4.10}$$

$$\bar{D} = D. \tag{4.11}$$

Corollary 4.5. Let $\bar{L}_i$ and L_i be i-pitches of $\bar{\varnothing}$ and $\varnothing$ resp. under the closed symmetric helical motion of order k. Then we have

$$\bar{L}_i = -L_i, \; 1 \leq i \leq m = k, \tag{4.12}$$

$$\bar{L} = L, \tag{4.13}$$

where $\bar{L} = \sqrt[m]{|\bar{L}_1 \ldots \bar{L}_m|}$ (pitch of $\bar{\varnothing}$).

Theorem 4.6. Let $\bar{\lambda}_i$ and λ_i be i-apex angles of $\bar{\varnothing}$ and $\varnothing$ resp. under the closed symmetric helical motion of order k. Then we have

$$\bar{\lambda}_i = -\lambda_i, \; 1 \leq i \leq m = k. \tag{4.14}$$

Proof: Since

$$\bar{\lambda}_i = \int_0^p \bar{w}_i(t) \, dt$$

and $\bar{w}_i = -w_i$, $1 \leq i \leq m = k$, we get

$$\bar{\lambda}_i = -\lambda_i, \; 1 \leq i \leq m = k.$$

Corollary 4.7. If $\overline{\lambda}$ and λ are apex angles of the axoids $\overline{\varnothing}$ and $\varnothing$ resp. under the symmetric helical motion of order k, then

$$\overline{\lambda} = \lambda.$$

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