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ON THE CLOSED PLANAR HOMOTHETIC MOTIONS AND THE ORBITAL AREAS

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ABSTRACT

In this study, it is shown that all the fixed points of the moving plane each of which has equal orbital areas in the fixed plane under the l-parameter closed planar homothetic motion lie on the same circle in the moving plane the center of which is generally different from the Steiner point. For the closed planar homothetic motions, obtaining a formula which is analogue of Hoditch formula some corollaries are found.

1. INTRODUCTION

1- Parameter Closed Planar Homothetic Motions

Let us consider 1–parameter homothetic motion of the moving plane E with respect to the fixed plane E′. We may consider the planes E and E′ as being complex planes. Hence each point X of the moving plane is to be observed as the representative of a complex number $x = x_1 + ix_2$ and the point X′ which is image of X under the homothetic motion is to be observed as the representative of a complex number $x' = x'_1 + ix'_2$. A 1–parameter homothetic motion of E with respect to E′ is given by

$$x' = h\, x\, e^{i\varnothing} + u' \tag{1}$$

where $h = h(t)$, $u' = u'(t)$ and $\varnothing = \varnothing(t)$ are functions of a real single parameter t and the complex number u′ represents the origin of moving system expressed in the system of the fixed plane. Let the complex number $u = u_1 + iu_2$ represents the origin of fixed system expressed in the system of moving plane. Then we have

$$u' = -ue^{i\varnothing} \ . \tag{2}$$

If T is the smallest positive number satisfying the following equalities

$$\left.\begin{aligned} u_j(t + T) &= u_j(t),\ j = 1, 2 \\ \varnothing(t + T) &= \varnothing(t) + 2\pi\nu \end{aligned}\right\} . \tag{3}$$

then the motion given by (1) is called 1–parameter closed planar homothetic motion with the period T and rotation number ν [1]. Such a motion is going to be shown by B. For the moving pole points of the motion B we have

$$\left.\begin{aligned} p_1 &= \frac{dh\,(du_1 - u_2\,d\varnothing) + hd\varnothing\,(du_2 + u_1\,d\varnothing)}{dh^2 + h^2\,d\varnothing^2} \\ p_2 &= \frac{dh\,(du_2 + u_1\,d\varnothing) - hd\varnothing\,(du_1 - u_2\,d\varnothing)}{dh^2 + h^2\,d\varnothing^2} \end{aligned}\right\} . \tag{4}$$

After some calculation, from (4), we obtain that

$$\left.\begin{aligned} du_1 &= p_1\,dh - p_2\,dh\varnothing + u_2\,d\varnothing \\ du_2 &= p_2\,dh + p_1\,hd\varnothing - u_1\,d\varnothing \end{aligned}\right\} . \tag{5}$$

or

$$\left.\begin{aligned} u_1 &= p_1 h + (p_2 dh)/d\varnothing - du_2/d\varnothing \\ u_2 &= p_2\,h - (p_1\,dh)/d\varnothing + du_1/d\varnothing \end{aligned}\right\} . \tag{6}$$

considering (1) and (5) we get

$$dx' = [(x_1 - p_1)\,dh - (x_2 - p_2)\,hd\varnothing] + i[(x_1 - p_1)\,hd\varnothing + (x_2 - p_2)dh] \tag{7}$$

II. On The Areas Enclosed By A Closed Curve Drawn By A Fixed Point of The Moving Plane Under The 1-Parameter Closed Planar Homothetic Motions

Let X be a fixed point in the moving plane E. then (1) defines a parametrized closed curve (X) in E′ which is called the trajectory curve of X under the motion B. The area enclosed by the closed curve (X) is

$$F_X = \tfrac{1}{2} \oint (x'_1 dx'_2 - x'_2 dx'_1) = \tfrac{1}{2} \oint [x', dx'] \tag{8}$$

where [a, b] is used instead of the determinant

$$[a, b] = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

and integrations are taken along the closed curve (X). Considering (7) and (8) we get

$$2F_X = (x^2{}_1 + x^2{}_2) \oint h^2\, d\varnothing - 2\, x_1 \oint p_1\, h^2\, d\varnothing - 2x_2 \oint p_2 h^2\, d\varnothing$$
$$+ x_1 \oint (u_2 dh - 2p_2 hah + hdu_2) + x_2 \oint (-u_1 dh + 2b_1 hdh - hdu_1)$$
$$+ \oint (u_1 p_1\, hd\varnothing + u_2 p_2\, hd\varnothing + u_1 p_2 dh - u_2 p_1 dh)\, , \tag{9}$$

Now, let us consider the Steiner point $s = s_1 + is_2$ of the moving centrode (P) for the distribution of mass with the density $h^2 d\varnothing$. Hence we have

$$s_i = \frac{\oint p_i h^2 d\varnothing}{\oint h^2 d\varnothing}\, , \ (i = 1{,}2) \tag{10}$$

where the integrations are taken along the closed curve (P). Let

$$\oint h^2\, d\varnothing = 2k\pi \tag{11}$$

where k is a real number such that $k = \nu$ only when $h = 1$. Then, from (9), (10) and (11) we obtain that

$$F_X = k\pi\, (x_1{}^2 + x^2{}_2 - 2s_1 x_1 - 2s_2 x_2) + F_0 + \tag{12}$$
$$+ 1/2x_1 \oint (-2p_2 hdh + hdu_2 + u_2 dh) + 1/2x_2 \oint (2p_1 hdh - hdu_1 - u_1 dh)$$

where F_0 is the area bounded by the trajectory curve of the origin O.

Theorem: All the fixed points of the moving plane which pass around equal areas under the motion B lie on the same circle in the moving plane. We can write

$$F_X = k\pi\, (x\bar{x} - \bar{x}\bar{s} - \bar{x}s) + F_0 + 1/2x_1 \oint (-2p_2 hdh + hdu_2 + u_2 dh) + 1/2x_2 \oint (2p_1 hdh - hdu_1 - u_1 dh) \tag{13}$$

where $\bar{x}$ and $\bar{s}$ are complex conjugates of x and s, respectively. As a result of (13) we can give the following corollary.

Corollary 1. If $h = 1$ then we have

$$\tfrac{1}{2} x_1 \int (-2p_2\, hdh + hdu_2 + u_2\, dh) + \tfrac{1}{2} x_2 \oint (2p_1\, hdh - hdh_1 - u_1\, dh) = 0.$$

So, in this special case (13) gives

$$F_X = \pi\nu\, (x\bar{x} - x\bar{s} - \bar{x}s) + F_0$$

which is the relation given by H.R. Müller [2].

III. On The Areas Enclosed By Closed Trajectory Curves Under The 1. Parameter Closed Planar Homothetic Motions And Holditch's Theorem

Let X and Y be two fixed points of the moving plane E and Z be another fixed point on the line segment $\overline{XY}$ that is, $z = \lambda x + \mu y$ where $\lambda + \mu = 1$. It can be easily seen that

$$z' = \lambda x' + \mu y' \ . \tag{14}$$

Now, we want to find the area enclosed by the trajectory curve (Z).

From (8) and (14) we can write

$$F_Z = 1/2 \oint [\lambda x' + \mu y', \lambda dx' + \mu dy'] \tag{15}$$

or

$$2F_Z = \lambda^2 F_X + \lambda\mu F_X Y + \mu^2 F_Y \tag{16}$$

where

$$F_{XY} = 1/2 \oint ([x', dy'] + [y', dx'])$$

and it is the mixture area of the trajectory curves (X) and (Y). Using (1), (4) and (7) in the last equation we obtain

$$\begin{aligned} 2F_{XY} = & - x_1 \oint p_1 h^2 \, d\varnothing - x_2 \oint p_2 h^2 \, d\varnothing + \\ & \oint (u_1 p_1 \, hd\varnothing + u_2 p_2 \, hd\varnothing + u_1 p_2 \, dh - u_2 p_1 \, dh) \\ & + x_1 \oint (u_2 dh - p_2 hdh) + x_2 \oint (p_1 hdh - u_1 \, dh) - y_1 \oint p_1 h^2 \, d\varnothing - y_2 \oint p_2 h^2 d\varnothing \\ & + \oint (u_1 p_1 hd\varnothing + u_2 p_2 \, hd\varnothing + u_1 p_2 dh - u_2 p_1 dh) + y_1 \oint (u_2 dh - p_2 hdh) \\ & + y_2 \oint (p_1 hdh - u_1 \, dh) + 2x_1 y_1 \oint h^2 \, d\varnothing + 2x_2 y_2 \oint h^2 d\varnothing - y_1 \oint u_1 hd\varnothing \\ & - y_2 \oint u_2 hd\varnothing - x_1 \oint u_1 \, hd\varnothing - x_2 \oint u_2 hd\varnothing \, . \end{aligned}$$

Considering (6), (13) and (17) we see that

$$F_{XY} = F_X + F_Y - k\pi d^2, \ d = |XY| \tag{18}$$

where $|XY|$ shows "the length of the line segment $\overline{XY}$". From (16) and (18) we find

$$F_Z = \lambda F_X + \mu F_Y - k\pi\lambda\mu \ d^2 \ . \tag{19}$$

On the other hand, since the motion does not depend on the choice of the co–ordinate systems we can chose the maving system such that the point X and Y lie on the real axis. Hence we find that

$$F_Z = \frac{1}{d}(aF_X + bF_Y) - k\pi\ ab \qquad (20)$$

where $a = y - z = \lambda d$, $b = z - x = \mu\ d$ and $d = a + b = y - x$. If the point X and Y draw the same convex trajectory curve under the motion B, then $F_X = F_Y$. Thus, from (20) we obtain

$$F_X - F_Z = k\pi\ ab\ . \qquad (21)$$

Hence we can give the following result.

Corollary 2. The area of the ring–shaped region between the trajectory curves (X) and (Z) depends on the homothetic motion because of k. If $h = 1$ then we have $k = \nu$. In this special case taking $\nu = 1$ it is obtained the result given by H.R. Müller [2].

Let us consider two different fixed points Z_1 and Z_2 on the line segment $\overline{XY}$ determined by the two fixed points X and Y of the moving plane. Let two of them, say X and Y move on the same curve (X) while the other two describe different curves (Z_1) and (Z_2). If the ring area between (X) and (Z_1) is F and the area between (X) and (Z_2) is F^1, then for the ratio F / F^1 we have

$$\frac{F}{F^1} = \left|\frac{|XZ_1|}{|XZ_2|}\right|^2 . \lambda \qquad \lambda = \frac{|XZ_2|}{|XZ_1|} \cdot \frac{|YZ_1|}{|YZ_2|}$$

where the magnitude λ is the cross ratio of the four points X, Y, Z_2 and Z_1, i.e. $\lambda = (XYZ_2Z_1)$. Hence we see that the ratio F / F^1 depends neither the curve (X) nor the length of the line segment $\overline{XY}$ but also only on the relative positions of these four points. So, we can give the following corollaries.

Corollary 3. Let us consider a 1–parameter closed planar homothetic motion and a fixed straight line k on the moving plane. And let us choose four arbitrary fixed points X, Y, Z_1 and Z_2 on the line k such that two of them draw the same curve (X) under the motion B while the other two describe different closed trajectory curves (Z_1) and (Z_2). If the ring area between (X) and (Z_1) is F and the area between (X) and (Z_2) is F^1, then the ratio F / F^1 depends only on the relative positions of these four points.

Corollary 4. Let X, Y, A and B be four points of the moving plane which pass around equal areas, and let the line segments $\overline{XY}$

and $\overline{AB}$ meet in Z. Then the necessary and sufficient condition for the points X, Y, A and B to lie on the same circle of the moving plane is

$$|XZ| . |YZ| = |AZ| . |BZ| \quad (22)$$

Proof: All the segments $\overline{XY}$, $\overline{AB}$, $\overline{XA}$, $\overline{YA}$, $\overline{XB}$ and $\overline{YB}$ are constant. Using (21) we get

$$F_A - F_Z = k\pi \ |AZ| . |BZ|$$

Comparison of (21) and (23) yields (22).

ÖZET

1- parametreli kapalı düzlemsel homotetik hareketler esnasında hareketli düzlemin, sabit düzlemde eşit alanlı yörünge eğrileri çizen bütün sabit noktaları hareketli düzlemde, merkezi Steiner noktası olmayan aynı bir çember üzerinde oldukları görüldü. Ayrıca homotetik hareketler için Holditch formülüne benzer bir bağıntı elde edilerek bazı sonuçlar verildi.

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