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LOCAL AND GLOBAL EXPOSED POINTS

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ABSTRACT

In this paper we derive sufficient conditions for strict convexity of subsets in a complete simply connected smooth Riemannian manifold without focal points in terms of local and global exposed points.

1. INTRODUCTION

In [12, 13], the concept of exposed points of subsets in linear metric spaces has been introduced. In [13], exposed points of subsets of Minkowski space have been specially considered. As far as I know the same concept and the corresponding local one have not yet been studied for subsets in curved spaces.

Hyperplanes as well as half-spaces bounded by hyperplanes were used in linear metric spaces to define and study exposed points. In a complete simply connected C^∞ Riemannian manifold W without conjugate points horospheres as well as horodiscs are the candidate geometric objects to play the same part of hyperplanes and half-spaces in linear spaces. Moreover, if we deal with subsets in a complete simply connected C^∞ Riemannian manifold W without focal points, horospheres behave nicely [7, 8]. Actually, in this case horospheres are level hypersurfaces of Bursemann functions. For certain Busemann function horospheres are equidistant (parallel) family of hypersurfaces whose orthogonal trajectories are geodesics [7, 8].

From now on let W (resp. $\tilde{W}$) denote a complete simply connected C^∞ Riemannian manifold without conjugate (resp. focal) points. Let W_p denote the tangent space of W at the point $p \in W$. $\bar{B}$ is the closure of the subset $B \subset W$. ∂B is the boundary of the subset B. All manifolds and maps are sufficiently smooth for discussions to make sense.

For each element v of the unit sphere bundle SW let $b_v: W \to R$ denote the Busemann function of v and H_v denote the horosphere of W characterized by v, i.e $H_v = \{x \in W: b_v(x) = 0\}$. D_v denotes the open horodisc of W, $D_v = \{x \in W: b_v(x) > 0\}$. Consequently $\partial D_v = H_v$. Notice that in Euclidean space E^n, b_v is the usual height function in the direction of v. For more details about the properties of horospheres and horodiscs see [7, 8, 10].

For basic properties of W and $\tilde{W}$ we refer the reader to [6, 7, 8, 9, 10]. The principal properties of convex subsets in Riemannian manifolds and metric spaces would be found in [1, 2, 6, 12, 13].

The following main facts concerning W and $\tilde{W}$ are needed throughout this work.

a) The exponential map $\exp_p: W_p \to W$ is a global diffeomorphism for each point $p \in W$.

b) Each manifold without focal points has no conjugate points but the converse is not generally true.

c) Each Riemannian manifold with sectional curvature $K \leq 0$ has no focal points [6].

d) For each pair of points $p, q \in W$ there exists a unique geodesic segment from p to q and will be denoted by [pq]. We shall write (pq) for the same segment with end points deleted. $\overrightarrow{pq}$ will denote the the geodesic ray through p and q with p as initial point.

In few words, the main goal of this paper is to derive sufficient conditions for strict convexity of subsets in $\tilde{W}$.

2. ON CONVEXITY

This section is mainly devoted to quote the following important facts concerning convexity.

A subset $B \subset W$ is convex if for each pair of points, $p, q \in B$ the connecting geodesic segment is contained in B. A convex subset with a non-empty interior is called a convex body. A convex subset $B \subset W$ is called strictly convex if its boundary ∂B contains no geodesic segments.

For an open convex subset $B \subset W$ we can show that the closure $\overline{B}$ is also convex (see [3, 4]). This is no longer valid in general Riemannian manifolds. It is direct to prove that the intersection of any number of convex subsets in W is convex taking into account that the empty set $\varnothing$ is itself convex.

Lemma 2.1. [3]

Geodesic balls in $\widetilde{W}$ are strictly convex bodies.

Lemma 2.2. [3]

Horodiscs in $\widetilde{W}$ are convex bodies.

Although horodisc in $\widetilde{W}$ is a limit of sequence of geodesic balls, horodisc is convex not necessarily strictly convex subset. Half-spaces in E^n are examples of this fact. Notice that convexity of geodesic balls and horodiscs in W is lost.

Lemma 2.3. [3]

Let $B \subset W$ be an open convex subset with smooth boundary ∂B and $p \in \partial B$ an arbitrary point. If γ is a maximal geodesic tangent to ∂B at p then $\gamma \cap B = \varnothing$. If B is strictly convex, then $\gamma \cap \overline{B} = \{p\}$.

From Lemma 2.2 and Lemma 2.3 we have the following

Corollary 2.4.

In $\widetilde{W}$ and for an arbitrary point $p \in \widetilde{W}$ let $\gamma: (-\infty, \infty) \to \widetilde{W}$ be maximal geodesic parametrized by arc-length such that $p = \gamma(0)$ and $\gamma'(0) = v \in W_p$. Then $\gamma \subset \widetilde{W} - (D_v \cup D_{-v})$.

The rest of this section will be devoted to give a short note about support element and its relation with convexity.

In [1], the support and local support elements in Riemannian manifold are given as follows.

If B is an open subset of a complete Riemannian manifold M, then an open half-space V_p of the tangent space M_p at $p \in \partial B$ is called a support element for B if V_p contains the initial tangent vectors of all minimal geodesics from p to points of B. V_p is a local support element for B if for some open neighborhood U of p, V_p is a support element for $B \cap U$.

Now we specialize to talk about support elements of subsets in manifolds without conjugate points.

Let us consider V_p to be a half-space of W_p. Using the exponential map $\exp_p: W_p \to W$, we have that $T(p) = \exp_p(\partial V_p)$ is a geodesic hypersurface of W at p since ∂V_p is an (n–1)–subspace of the linear space W_p, $\dim W_p = n$. As $\exp_p$ is a global diffeomorphism then W–T(p) has two unbounded components one of them is the image of V_p under $\exp_p$.

Let B be an open subset of W such that B is contained in one component of W–T(p) for some $p \in \partial B$. Using $\exp_p^{-1}: W \to W_p$ we have that $\exp_p^{-1} B$ is contained in an open half–space, say V_p, of W_p. Consequently for any arbitrary point $x \in B$ the unique connecting geodesic segment [px] from p to x has the property that $\exp_p^{-1}(px) \subset V_p$ which ensures that the initial velocity of [px] is contained in V_p.

From the above argument we obtain the following information about support elements in W.

Let B be an open subset of W and $p \in \partial B$. B has a support element at p if and only if B is contained wholly in a component of W–T(p) for some geodesic hypersurface T(p) at p. B has a local support element at p if and only if there exists a neighborhood U about p in W such that $B \cap U$ is contained in a component of W–T(p). Moreover, if ∂B is a smooth hypersurface of W then T(p) will be tangent to ∂B at p. In this case T(p) is unique.

In the light of the above mentioned discussion, Proposition (1) [1] may be restated in W as follows.

Proposition 2.4.

A connected open subset B of W is convex if and only if B has the property that at each boundary point $p \in \partial B$ there exists a neighborhood U of p such that $B \cap U$ is contained in one component of W–T(p) where T(p) is a geodesic hypersurface at p.

3. MAIN RESULTS

Definition 3.1. For a subset $B \subset W$ the point $p \in B$ is an exposed point of B if there exists a unit vector $v \in W_p$ such that

(i) $B \subset \overline{D}_v$ (ii) $B \cap H_v = \{p\}$.

Definition 3.2. For a subset $B \subset W$ the point $p \in B$ is called a local exposed point of B if there exists a neighborhood U of W about p such that $B \cap U$ has p as an exposed point.

From the above definitions it becomes clear that for a subset $B \subset W$, no interior point could be either exposed or local exposed point. Each global exposed point is local exposed but the converse is not generally true. One can show that if B is a compact subset of $\tilde{W}$, then B has at least two exposed points [5]. Moreover if a subset $B \subset W$ has an exposed point $p \in B$, then there exists a unit vector $v \in W_p$ such that $b_v(x) > 0$ for all $x \in B$, $x \neq p$.

Definition 3.3. Let $B \subset W$ be a subset. A point $p \in W$ has a foot point q in B if

(i) $q \in B$ (ii) $d(p, q) = d(p, B)$

where d(p,B) denotes the Hausdorff distance from p to B.

A point $p \in B$ is a foot of itself in B. For a compact subset $B \subset W$ a point $p \in B$ has a foot point in B [12]. For other properties of foot points see [4].

In this section we establish –as main results– the following two theorems 3.4 and 3.5 which relate convexity of subsets of $\tilde{W}$ with both local and global exposed points of the same subset.

Theorem 3.4. Let $B \subset \tilde{W}$ be a connected open subset with smooth boundary ∂B. Assume that each boundary point is a local exposed point of $\bar{B}$, then B is a strictly convex body of $\tilde{W}$.

Proof: Let us consider an arbitrary point $p \in \partial B$. Let $\gamma: [0, \infty) \to \tilde{W}$ be the interior geodesic ray parametrized by arc–length such that $\gamma(0) = p$ and γ is starting perpendicular to ∂B. Let us write $\gamma'(0) = v$. As p is a local exposed point of $\bar{B}$, then there exists a neighborhood U about p such that p is an exposed point of $\bar{B} \cap U$. Hence $\bar{B} \cap U \subset \bar{D}_v$ and $(\bar{B} \cap U) \cap H_v = \{p\}$ (see Fig. (1)). Using Corollary 2.4 we have that $B \cap U$ is contained in one component of $\tilde{W} - T(p)$ where T(p) is the geodesic hypersurface of $\tilde{W}$ at p tangent to ∂B at p. Hence $B \cap U$ has a support element at p which is a local support element of B at p. Applying Proposition 2.4 and taking into account that p is an arbitrary point of ∂B we have that B is a convex body of $\tilde{W}$.

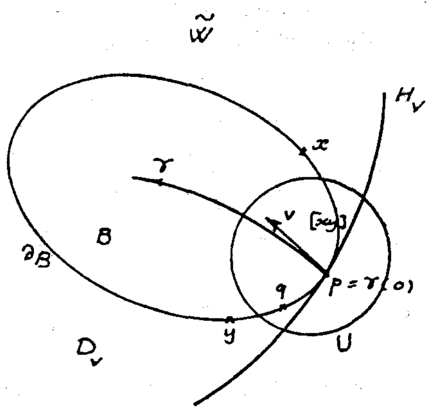

Fig. (1)

It remains now to show that B is strictly convex.

Assume in contrary that B is convex but not strictly convex. Consequently, there exists a geodesic segment $[xy] \subset \partial B$ joining some pair of points x, $y \in \partial B$. Without loss of generality assume that the point p mentioned above is the mid-point on $[xy]$ between x and y. Let $q \in (\overline{B} \cap U) \cap [xy]$ and $q \neq p$ (see Fig. (1)). Since p is a local exposed point of $\overline{B}$, then the geodesic ray $\overrightarrow{pq}$ satisfies $q \in \overrightarrow{pq} \cap D_v$, i.e $\overrightarrow{pq} \cap D_v \neq \varnothing$ contradicting Lemma 2.3, and the proof of Theorem 3.4 is now completed.

It is worth mentioning that the above Theorem 3.4 could be proved through showing that $\overline{B}$ is the intersection of a family of closed horodiscs in $\tilde{W}$.

Theorem 3.5. Let $B \subset \tilde{W}$ be an open bounded subset with smooth boundary ∂B. Assume that each boundary point is a global exposed point of $\overline{B}$, Then B is a strictly convex body of $\tilde{W}$.

Proof: The crucial point in the proof is to show that B is a connected subset of $\tilde{W}$. Assume in contrary that B is disconnected and assume without loss of generality that $B = B_1 \cup B_2$ where B_1 and B_2 are disjoint open bounded subsets of $\tilde{W}$. Consider an arbitrary point $p \in B_1$.

Since $\overline{B}_2$ is compact and $p \notin B_2$, then there exists a point $q \in \overline{B}_2$ which is a foot point of p in $\overline{B}_2$. The geodesic segment [pq] meets ∂B_2 orthogonally (transversally) at q and so there exists a point, say m, in B_2 such that $[pq] \subset [pm]$, i.e q lies between p and m on the geodesic segment [pm] (see Fig. (2)). Let $v \in \tilde{W}_q$ be the velocity of [pm] at q and assume that v is a unit vector. Considering the closed horodiscs $\overline{D}_v$ and $\overline{D}_{-v}$ we have that for the points p, $m \in B$,

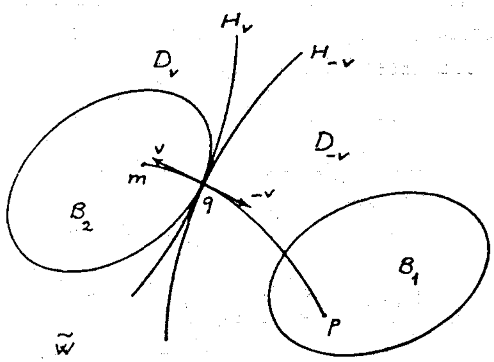

Fig. (2)

$$b_v(m) > 0, \qquad b_v(p) < 0$$
$$b_{-v}(m) < 0, \qquad b_{-v}(p) > 0$$

which means that $q \in \partial B$ is not an exposed point of $\overline{B}$ contradicting the assumption of the theorem. Hence B is connected.

Since each boundary point $p \in \partial B$ is an exposed point of $\overline{B}$, then p is a local exposed point of $\overline{B}$. Now we can repeat the same disscussion mentioned in the proof of Theorem 3.4 to show that B is a strictly convex body of $\tilde{W}$.

Remarks 3.6. (a) Neither the converse of Theorem 3.4 nor that of Theorem 3.5 is generally true. We give here only an example in the 2-dimensional hyperbolic space H^2 in its half-space model to ensure our claim concerning Theorem 3.4.

Let H^2 be the subset $\{(x, y) : y > 0\} \subset E^2$ under the hyperbolic metric [11]. Let $B \subset H^2$ be the open subset

$$B = B(p, a + \delta) \cap H^2$$

where B (p, a + δ) is an Euclidean open ball centered at p $= (0, a-\delta)$ $a > 0$, with radius $a + \delta$ for sufficiently small $\delta > 0$. We can see easily that B is a strictly convex body in H^2 while the point (0, 2a) is not a local exposed point of $\bar{B}$. The closed horodiscs $\overline{D}_V$ and $\overline{D}_{-V}$ for the unit vector $v \in H^2{}_{(0,2a)}$ are indicated in Fig. (3). Notice that H_V is an Euclidean circle S (q, a) where q $= (0, a)$ while H_{-V} is the Euclidean straight line $y = 2a$. (b) In hyperbolic space H^n, we can define two different types of exposed points in regard to horospheres or totally geodesic hypersurfaces. In the last case when adopting the totally geodesic hypersurfaces idea, we can prove that Theorems 3.4 and 3.5 and their converses are valid.

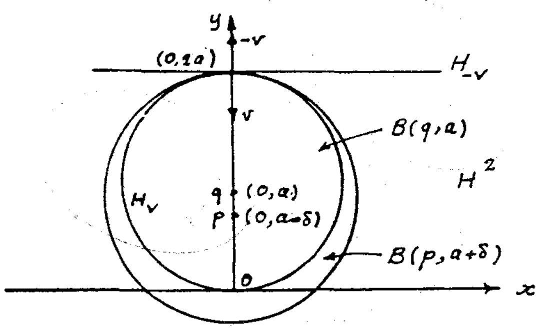

Fig. (3)

c) We can study exposed points of subsets in sphere S^n as a manifold with focal (or conjugate) points in terms of totally geodesic hypersurfaces. Cut points (or antipodal points) should be taken into account in this case.

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