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VERTICAL AND COMPLETE LIFTS ON STATISTICAL MANIFOLDS AND ON THE UNIVARITE GAUSSIAN MANIFOLDS

MURATHAN, C., GÜNEY, İ.

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ABSTRACT

In this article geometrical analysis of geometric inference problems have developed by means of differential geometrical methods, we give vertical and complete lifts of curvature tensors, connections etc. to the tangent bundles of Statistical manifolds and Univariate Gaussian Manifolds, also. Furthermore, we give necessary and sufficent conditions for ellipcity of hyperbolocity of TM.

INTRODUCTION

A statistical model often forms a geometrical manifold, so that the geometry of manifold should play an important role. Considering that properties of spesific types of probability distributions, for example, of Gaussian distributions, have so far been studied in detail, it seems rather strange that only a few theories have been proposed concerning properties of a family itself of distributions. Here, by the properties of a family we mean such geometric relations as mutual distances, flatness or curvature of the family etc.

Let $S=\{p(x,\Theta)\}$ be a statistical model consisting of probability density functions $p(x,\Theta)$ of random variable $x \in X$ with respect to a measure P on X such that every distribution is uniquely parametrized by an n-dimensional vector parameter $\Theta=(\Theta^1,\ldots,\Theta^n)$.

A statistical model S is said to be geometrically regular, when it satisfies the following conditions A_1—A_6.

A1–The domain θ of the parameter Θ is homeomorfic to an n-dimensional Euclidian space R^n.

* Department of Mathematics Uludağ University Art and Sci. Fac. 16059 Görükle-Bursa.
** Department of Mathematics Uludağ University Art and Sci., Fac. 16059 Görükle-Bursa.

A2–The topology of S induced from R^n is compatible with the relative topology of S in L_1 space.

A3–The support of $\rho(x,\Theta)$ is common for all $\Theta\in\theta$, so that $p(x,\theta)$ are mutually absolutely continuous.

A4–Every density function $p(x,\theta)$ is a smooth function in Θ unifomly in x, and the partial derivative ∂/∂^i and integration of log $p(x,\Theta)$ with respect to the measure P(x) are always commutative.

A5–The moment of score function (∂/∂^i) log $p(x,\Theta)$ exist up to the third order and are smooth in Θ.

A6–The Fisher information matrix is positive definite.

I. VERTICAL AND COMPLETE LIFTS ON STATISTICAL MANIFOLDS

Let (M,g) be a n- dimensional Riemannian manifold and denote the (m,n)-tensor fields on M by $J_n^m(M)$. We call a triple (M,g,D) is a statistical manifold if (M,g) is a Riemann manifold and the tensor field $D\in J^0_3$ (M) which satisfies the following;

$$D(X,Y,Z)=D(Y,Z,X)=D(X,Z,Y)=D(Y,X,Z)=D(Z,X,Y)=D(Z,Y,X)$$

for all X,Y,Z $\in J^1_0(M)$, [2]. The tensor field D is called the skewness of the manifold (M,g,D), [2]. On a statistical manifold (M,g,D) there exists a uniqe tensor field $D^\sim \in J^1_2(M)$ such that

$$g(D^\sim(X,Y),Z) = D(X,Y,Z) \tag{1.1}$$

for all X,Y,Z $\in J^1_0(M)$, [2]. We now define a family of connections as follows:

$$\nabla^\alpha_X Y = \nabla_X Y - \frac{\alpha}{2} D^\sim (X,Y), \ \forall X,Y\in J^1_0(M) \tag{1.2}$$

where ∇ is the Riemannian connection on (M,g) and the α is a real number. Also ∇^α is a torsion free connection on (M,g), [2]. Let $\{x^i\}$ be a local coordinate system on M and consider the following tensor fields on M

$$D^\sim = \Sigma D^h_{ji} \frac{\partial}{\partial x^h} \otimes dx^j \otimes dx^i : J^1_0(M)xJ^1_0(M) \to J^1_0(M), \tag{1.3}$$

$$D = \Sigma D_{jik} dx^j \otimes dx^i \otimes dx^k : J^1_0(M)xJ^1_0(M)xJ^1_0(M) \to C^\infty(M). \tag{1.4}$$

We denote complete lift of $D^{\sim}$ and D by $D^{\sim c}$ and D^c, respectively, so we have;

$$D^{\sim c} = \Sigma\ [\partial D^h{}_{ji} \frac{\partial}{\partial y^h} \otimes dx^j \otimes dx^i + D^h{}_{ji} \frac{\partial}{\partial x^h} \otimes dx^j \otimes dx^i +$$

$$D^h{}_{ji} \frac{\partial}{\partial y^h} \otimes dy^j \otimes dx^i + D^h{}_{ji} \frac{\partial}{\partial y^h} \otimes dx^j \otimes dy^i] \qquad (1.5)$$

and

$$D^c = \Sigma\ [\partial D_{jik} dx^j \otimes dx^i \otimes dx^k + D_{jik} dy^j \otimes dx^i \otimes dx^k +$$
$$D_{jik} dx^j \otimes dy^i \otimes dx^k + D_{jik} dx^j \otimes dx^i \otimes dy^k] \qquad (1.6)$$

where $\{x^h, y^h\}$ is the induced local coordinate system on TM from M. If ∇^{α}, ∇ and $D^{\sim}$ have components $Q^h{}_{ji}, \Gamma^h{}_{ji}$ and $D^h{}_{ji}$, respectively, then

$$Q^h{}_{ji} = \Gamma^h{}_{ji} - \frac{\alpha}{2} D^h{}_{ji} \qquad (1.7)$$

can be written from [2]. Thus we have

Lemma 1.

$$D^{\sim c}(X^c, Y^c) = (D^{\sim}(X,Y))^c \qquad (1.8)$$

for any X, $Y \in J_0{}^1(M)$.

Lemma 2. If the components of $D^{\sim c}$ are denoted by $D^{-A}{}_{CB}$ then

$$\left.\begin{aligned} & D^{-h}{}_{ji} = D^h{}_{ji},\ D^{-h}{}_{\bar{j}i} = 0,\ D^{-h}{}_{j\bar{i}} = 0,\ D^{-h}{}_{\bar{j}\bar{i}} = 0 \\ & D^{-\bar{h}}{}_{ji} = \partial D^h{}_{ji},\ D^{-\bar{h}}{}_{\bar{j}i} = D^h{}_{ji},\ D^{-\bar{h}}{}_{j\bar{i}} = D^h{}_{ji},\ D^{-\bar{h}}{}_{\bar{j}\bar{i}} = 0 \end{aligned}\right\} \qquad (1.9)$$

where $D^h{}_{ji}$ are the components of $D^{\sim}$.

Lemma 3.

$$D^{\sim c}(X^v, Y^v) = 0,\ D^{\sim c}(X, Y^v) = (D^{\sim}(X,Y))^v \qquad (1.10)$$
$$D^{\sim c}(X^v, Y^c) = (D^{\sim}(X,Y))^v$$

for all X, $Y \in J^1{}_0(M)$.

Theorem 1. If the complete lifts of the connections ∇^{α}, ∇, and D^- to the tangent bundle TM are $\nabla^{\alpha c}$, ∇^c and D^c, respectively, then

$$\nabla^{\alpha c}{}_{X^c} Y^c = (\nabla_X{}^{\alpha} Y)^c$$

for all $X, Y \in J^1{}_0(M)$.

Theorem 2. If the components of $\nabla^{c\alpha}$ is $Q\sim^A{}_{CB}$ then

$$Q\sim^h{}_{ji} = Q^h{}_{ji},\ Q\sim^h{}_{\bar{j}i} = 0,\ Q\sim^h{}_{j\bar{i}} = 0,\ Q\sim^h{}_{\bar{j}\bar{i}} = 0$$

$$Q\sim^{\bar{h}}{}_{ji} = \partial\Gamma^h{}_{ji} - \alpha/2\ \partial(D^h{}_{ji}),\ Q\sim^{\bar{h}}{}_{\bar{j}i} = Q^h{}_{ji} \qquad (1.11)$$

$$Q\sim^{\bar{h}}{}_{j\bar{i}} = Q^h{}_{ji},\ Q\sim^{\bar{h}}{}_{\bar{j}\bar{i}} = 0,$$

where,

$$Q^h{}_{ji} = \Gamma^h{}_{ji} - \alpha/2\ D^h{}_{ji}$$

and $\Gamma^h{}_{ji}$, $D^h{}_{ji}$ are the components of ∇ and $D\sim$, respectively.

Corollary 1. Let R^α be the curvature of ∇^α and the components of $\triangle^\alpha$ and R^α are $Q^h{}_{ji}$, $R^{\alpha h}{}_{kji}$, respectively, then

$$R^{\alpha h}{}_{kji} = \partial_k(Q^h{}_{ji}) - \partial_j(Q^h{}_{ki}) + Q^h{}_{kt}\, Q^t{}_{ji} - Q^h{}_{kt}\, Q^t{}_{ji}.$$

Corollary 2. If the components of R^α are $R^{\alpha h}{}_{kji}$ and the components of R are $R^h{}_{kji}$ then

$$R^{\alpha h}{}_{kji} = R^h{}_{kji}.$$

Proof:

$$R^{\alpha h}{}_{kji} = \partial_k(\Gamma^h{}_{ji} - \frac{\alpha}{2} D^h{}_{ji}) - \partial_j(\Gamma^h{}_{ki} - \frac{\alpha}{2} D^h{}_{ki}) +$$

$$(\Gamma^h{}_{kt} - \frac{\alpha}{2} D^h{}_{kt})\ (\Gamma^t{}_{ji} - \frac{\alpha}{2} D^t{}_{ji})$$

$$- (\Gamma^h{}_{jt} - \frac{\alpha}{2} D^h{}_{jt})\ (\Gamma^t{}_{ki} - \frac{\alpha}{2} D^t{}_{ki})$$

$$= R^h{}_{kji} + \frac{\alpha}{2}\ (\partial_j(D^h{}_{ki}) - \partial_k(D^h{}_{ji})) + \frac{\alpha}{2}\ (\Gamma^h{}_{jt}D^t{}_{ki} - \Gamma^h{}_{kt}D^t{}_{ji})$$

$$+ \frac{\alpha}{2}(\Gamma^t{}_{ki}D^h{}_{jt} - \Gamma^t{}_{ji}D^h{}_{kt}) + \frac{\alpha^2}{4}\ (D^h{}_{kt}D^t{}_{ji} - D^h{}_{jt}D^t{}_{ki})$$

$$= R^h{}_{kji}$$

as desired.

Corollary 3. If the curvature tensor of M with respect to the connection ∇^α and ∇ are R^α and R, respectively, then

$$R^\alpha = R,\ R^{\alpha c} = R^c$$

Corollary 4. If the components of $R^{\alpha c}$ are $R^{\sim\alpha A}{}_{DCB}$, the components of R^c are $R^{\sim A}{}_{DC}{}^{s}{}_{B}$ then

$$R^{\sim\alpha h}{}_{kji} = R^h{}_{kji},\ R^{\sim\alpha\bar{h}}{}_{kji} = \partial R^h{}_{kji},$$

$$R^{\sim\alpha\bar{h}}{}_{kj\gamma} = R^h{}_{kji},\ R^{\sim\bar{\alpha h}}{}_{k\bar{j}i} = R^h{}_{kji},\ R^{\sim\alpha\bar{h}}{}_{\bar{k}ji} = R^h{}_{kji}.$$

Theorem 3. Letting L_X be Lie derivative on M with respect to $X \in J^1{}_0$ (M)

$$(L_X)\nabla^{\alpha}(Y,Z) = (L_X)\nabla(Y,Z) - \frac{\alpha}{2}(L_X)D^{\sim}(Y,Z)$$

for all $Y,Z \in J^1{}_0(M)$.

Proof: By applying Lie differantion on $\nabla^{\alpha}(Y,Z)$, we can wite

$$(L_X)\nabla^{\alpha}(Y,Z) = (L_X)\nabla^{\alpha}{}_Y Z - \nabla^{\alpha}{}_Y(L_X Z) - \nabla^{\alpha}{}_{[X,Y]}Z$$

$$= L_X(\nabla_Y Z - \frac{\alpha}{2}D^{\sim}(Y,Z)) - \nabla_Y(L_X Z) + \frac{\alpha}{2}D^{\sim}(Y,L_X Z) - \nabla_{[X,Y]}Z$$

$$+ \frac{\alpha}{2}D^{\sim}([X,Y],Z) = (L_X\nabla)(Y,Z) - \frac{\alpha}{2}(L_X D^{\sim})(Y,Z)$$

that was what we are looking for.

Following two theorems are easy adoptations from [1];

Theorem 4.

$$\nabla^{\alpha c}{}_{x^v} f^v = 0,\ \nabla^{\alpha c}{}_{x^v} f^c = (\nabla^{\alpha}{}_x f)^v,$$

$$\nabla^{\alpha c}{}_{x^c} f^v = (\nabla^{\alpha}{}_x f)^v,\ \nabla^{\alpha c}{}_x f^c = (\nabla^{\alpha}{}_x f)^v$$

for all $X \in J^1{}_0(M)$ and $f \in J^0{}_0(M)$.

As shown in [1], if ∇ is the Riemannian connection then we have

$$\nabla^c{}_{x^v} Y^v = 0,\ \nabla^c{}_{x^v} Y^c = (\nabla_x Y)^v \quad (1.12)$$

$$\triangle^c{}_{x^c} Y^v = (\Delta_x Y)^v,\ \triangle^c{}_{x^c} Y^c = (\triangle_x Y)^c \quad (1.13)$$

for all $X, Y \in J^1{}_0(M)$.

Thus we can give the following theorems;

Theorem 5. By the notations used as so far, then

$$\nabla^{\alpha c}{}_{x^v} Y^v = 0,\ (\nabla^{\alpha c}{}_{x^v} Y^c = (\nabla^{\alpha x} Y)^v,$$

$$\nabla^{\alpha c}{}_{x^c} Y^v = (\nabla^{\alpha}{}_x Y)^v,\ \nabla^{\alpha c}{}_{x^c} Y^c = (\nabla^{\alpha}{}_x Y)^c$$

for all $X,Y \in J^1{}_0(PM)$.

Theorem 6. By the notations used as so far

$$\nabla^{\alpha c}_{\ X^v} W^v = 0 \ , \ \nabla^{\alpha c}_{\ X^v} W^c = (\nabla^{\alpha}_{\ X} W)^v \ ,$$

$$\nabla^{\alpha c}_{\ X^c} W^v = (\nabla^{\alpha}_{\ X} W)^v \ , \ \nabla^{\alpha c}_{\ X^c} W^c = (\nabla^{\alpha}_{\ X} W)^c$$

for all X, Y in $J^1_0(M)$ and W in $J_1^0(M)$.

II. THE VERTICAL AND COMPLETE LIFTS OF UNIVARIATE GAUSS MANIFOLDS

Let us consider the family of normal distributions $N(\mu,\sigma^1)$, i.e. the family with densities

$$f(x,\mu,\sigma) = (2\pi\sigma^2)^{-1/2} \exp(-(2\sigma^2)^{-1}(x-\mu)^2\}; \ \mu \in R, \sigma > 0 \tag{2.1}$$

Working in the (μ,σ) parametrization we obtain the following expression for the g- metric, the α-connections and D tensor (skewness) expressed in D_{ijk} [2].

Consider the metric $g = \Sigma_{j,i}\, g_{ji} dx^j \otimes dx^i$ such that the compotents g_{ji} are satisfies the following

$$g_{11} = (1/\sigma^2), \ g_{22} = (2/\sigma^2), \ g_{12} = g_{21} = 0. \tag{2.2}$$

Assume further that: for components Q_{jik} of α- connection and the components D_{jik} of the tensor D hold the following equations

$$Q_{111} = Q_{122} = Q_{212} = Q_{221} = 0$$

$$Q_{112} = (1-\alpha)/\sigma^3, \ Q_{121} = Q_{211} = -(1+\alpha)/\sigma^3 \tag{2.3}$$

$$Q^1_{11} = Q^2_{12} = Q^2_{21} = Q^1_{22} = 0$$

$$Q^1_{12} = Q^2_{21} = -(1+\alpha)/\sigma, \ Q^2_{22} = -(1+2\alpha)/\sigma \tag{2.4}$$

$$D_{111} = D_{122} = D_{212} = D_{221} = 0$$

$$D_{112} = D_{121} = D_{211} = 2/\sigma^3 \ D_{222} = 8/\sigma^3 \ . \tag{2.5}$$

In addition the α- curvature tensor is given by

$$R^{\alpha}_{1212} = (1-\alpha^2)/\sigma^4 \tag{2.6}$$

and also the scalar (sectional) curvature is given by

$$K^{\alpha}(\sigma_{12}) = R^{\alpha}_{1212}/(g_{11}-g_{22}) = -(1-\alpha^2)/\ 2 \tag{2.7}$$

[2].

Let g^c be the complete lift of the α- metric g and $(\mu.\sigma)$ denotes a coordinate neighborhood of M, also, the induced coordinate neighborhood (μ,σ,y^1,y^2) of TM then

$$\partial g_{11} = -2y^2/\sigma^3,\ \partial g_{12} = \partial g_{21} = 0,\ \partial g_{22} = -4y^2/\sigma^3, \tag{2.8}$$

$$g_{11} = (1/\sigma^2),\ g_{22} = (2/\sigma^2),\ g_{12} = g_{21} = 0$$

$$Q^c{}_{111} = Q^c{}_{122} = Q^c{}_{212} = Q^c{}_{221} = 0 \tag{2.9}$$

$$Q^c{}_{112} = 3y^2(\alpha - 1)/\sigma^4,\ Q^c{}_{121} = Q^c{}_{211} = 3y^2(1+\alpha)/\sigma^4$$

$$Q^{c1}{}_{11} = Q^{c2}{}_{12} = Q^{c2}{}_{21} = Q^{c1}{}_{22} = 0 \tag{2.10}$$

$$Q^{c1}{}_{12} = Q^{c1}{}_{21} = y^2(1+\alpha)/\sigma^2,\ Q^{c2}{}_{22} = y^2(1+2\alpha)/\sigma^2$$

$$D^c{}_{111} = D^c{}_{122} = D^c{}_{212} = D^c{}_{221} = 0 \tag{2.11}$$

$$D^c{}_{112} = D^c{}_{121} = D^c{}_{211} = -6y^2/\sigma^4,\ D^c{}_{222} = -24y^2/\sigma^4$$

and the α- curvature tensor of TM has the components

$$R^{\alpha c}{}_{1212} = 4y^2(\alpha^2 - 1)/\sigma^5 \tag{2.12}$$

and for scalar curvature we have

$$K^{\alpha c}(\sigma_{12}) = 0\ . \tag{2.13}$$

Theorem 2.1. Let (μ, σ, y^1, y^2) be a coordinate neighboorhood of TM, then we have

1) If y^2 is positive then
   - i) TM is elliptic iff M is hyperbolic.
   - ii) TM is hyperbolic iff M is elliptic.
2) If y^2 is negative then
   - iii) TM is elliptic iff M is elliptic.
   - iv) TM is hyperbolic iff M is hyperbolic.
3) If $y^2 \neq 0$

   M is flat iff TM is flat.

Notice that, $y^2 = 0$, then TM is flat but M does not to be flat.

Proof: From (2.6) and (2.12) we have

TM is elliptic iff i1) y^2 is positive and α in $(-\infty, -1)$ U $(1, \infty)$,

i2) y^2 is negative and α in $(-1, 1)$.

TM is hyperbolic iff i3) y^2 is negative and α in $(-\infty, -1) \cup (1, \infty)$,

i4) y^2 is positive and α in $(-1, 1)$.

TM is flat iff $y^2 = 0$ or $\alpha = -1$ or $\alpha = 1$.

Also,

a) M is elliptic iff α in $(-1, 1)$,

b) M is hyperbolic iff α in $(\infty-, -1) \cup (1, \infty)$,

c) M is flat iff either $\alpha = -1$ or $\alpha = 1$

which completes the proof of the theorem.

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