

ON THE INVARIANCE OF BAIRE SPACES UNDER THE VARIOUS TYPES OF MAPPINGS

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ABSTRACT

A mapping from a space X into a space Y is called δ -open if the image of each somewhere dense subset of X is a somewhere dense subset of Y , or equivalently for every nowhere dense subset N of Y , $f^{-1}(N)$ is nowhere dense subset of X [6. p.45]. In this paper, we first consider a proposition which was proved in [6. p.45] on δ -open mappings. We improve this proposition and extend it to the various types of mappings and some results which show that the preimage of a Baire space is Baire space under the various types of bijection mappings, are obtained.

INTRODUCTION

Recent mathematical literature contains different weakened forms of continuity such as weak-continuity, semi-continuity, almost continuity etc. In 1961 N. Levine [8] introduced weak-continuity and in 1963 he [9] defined semi-continuity as a weakened form of continuity. On the other hand, in 1966 T. Husain [7] defined almost-continuity which is another weakened form of continuity and the concept of almost-open mapping was introduced in 1967 by A. Wilansky [22]. But in 1968. M.S. Singal and A.R. Singal [18] defined almost-continuous mappings, almost-open mappings in different forms. Moreover. M. Biswas [2] introduced semi-open mappings which contain the class of open mappings and in 1984 D.A. Rose [17] defined weakly-open mappings as a new weakened form of open mappings.

This paper is divided into two sections. The first section begins with some properties of semi-open mappings, nearly feebly-continuous mappings [16] which we need in sequel. Then, we give the examples which show that neither of two forms of almost-open mappings, in general, imply other. Furthermore, a fundamental relation (see, Theorem 1.9 of section 1) between these two almost-open mappings is established by the notion of semi-continuity. This relation is then used to prove a basic theorem (see, Theorem 2.1 of section 2). The ex-

amples which show that the feebly-open mappings [5] and two forms of almost-open mappings are independent of each other, are given in the same section. In addition, we shall establish some relations between weakly-open mappings and two forms of almost-open mappings. These relations will be used frequently throughout the sequel. After that the examples which show that the feebly-continuous mappings [5] and two forms of almost-continuous mappings are independent of each other are given in this section. As an improvement of [10. Theorem 5], we establish the other relation between two forms of almost-continuity (see, Theorem 1.17 of section 1). This relation is then used to prove a Lemma of the second section. The first section ends with some results which show the relationships between weak-continuity and two forms of almost-continuity.

R.C. Howorth and R.A. McCoy [6, p. 45] introduced the notion on δ -open mappings and they proved a proposition [6, proposition 4.4, p. 45] on δ -open mappings. In the second section, we begin extending this proposition to the various classes of mappings. Here we consider mappings $f: X \rightarrow Y$ which are;

- 1) semi-continuous, nearly feebly open (i.e. $\text{int}(f(\overline{U})) \neq \emptyset$ for each non empty open set U in X).
- 2) semi-continuous and almost-open in the sense of Wilansky.
- 3) semi-continuous and almost-open in the sense of Singal and Singal.
- 4) semi-continuous and irreducible semi-closed.
- 5) semi-continuous and semi-open.
- 6) semi-continuous and feebly-open.

In 1961. Z. Frolik [5. p. 383] proved that if f is an almost continuous and feebly open mapping of a Baire space X onto a space Y , then Y is a Baire space. We recall the notion of almost-continuity in this theorem is known in the literature as "semi-continuity". Therefore, we use the notion of "semi-continuity". In the second section it is shown that Frolik's Theorem can be extended to the first five types of mappings above mentioned. Furthermore, the sufficient conditions which insure that the image of a Baire space is a Baire space under the various types of surjection mappings, are given in the same section (see, Corollary 2.3-Corollary 2.8 of section 2).

One may ask that whether the inverse image of a Baire space must be Baire space under semi-continuous and nearly feebly-open bijection mapping. The answer of this question is not positive in general (e.g., see Example 1.2 of section 1). However we establish that if f is one-to-one semi-open and nearly feebly-continuous (i.e, $\text{int}(\overline{f^{-1}(V)}) \neq \emptyset$ for non empty open V in Y) mapping from space X onto a Baire space Y then X is a Baire space (see, Theorem 2.9 of section 2). The remainder of second section is concerned with several results which show that the preimage of a Baire space is a Baire space under the various types of bijection mappings (see Corollary 2.11-Corollary 2.20 of section 2)

The most frequently used notations are following; Let A be a subset of a topological space X . The closure of A in X and interior of A in X will be denoted by \bar{A} and $\text{int} A$ respectively. The complement of A in X is $X-A$. Throughout this paper X and Y will be always denote topological spaces on which no separation axiom are assumed unless stated explicitly. No mapping is assumed to be continuous unless stated.

1. DEFINITIONS AND PRELIMINARY RESULTS

In this section, various types of mappings are studied and the background results are given in the same section.

Definition 1.1. Let A be a subset of X . A is said to be a semi-open [9], if there exists an open subset O of X such that $O \subset A \subset \bar{O}$.

The following result can be found in [9].

Theorem 1.1. A subset A in X is semi-open if and only if $A \subset \text{int} \bar{A}$

Definition 1.2. A mapping $f: X \rightarrow Y$ is said to be semi-continuous [9] (resp., semi-open [2]) if for each open subset V in Y (resp., open subset U in X), $f^{-1}(V)$ is semi-open in X (resp. $f(U)$ is semi-open in Y).

Remark 1.1. Of course, any continuous (resp., open) mapping is semi-continuous (resp. semi-open), but the converse of these statements is not true as is shown by the following example.

Example 1.1. Let \mathcal{U} be usual topology on the real line and \mathcal{J} be lower limit topology on the real line which is generated by the right half-open intervals $[a, b)$, $a, b \in \mathbb{R}$. Furthermore let i be the identity

mapping from (R, \mathcal{I}) onto (R, \mathcal{F}) . Then i is semi-continuous, but it is not continuous. On the other hand, inverse of identity mapping i^{-1} is semi-open, but it is not open.

We have the following results which will be needed for our proof in sequel.

Theorem 1.2. Let $f: X \rightarrow Y$ be semi-open bijection mapping, then $\text{int}(\overline{f(A)}) \subset f(\bar{A})$, for each subset A of X .

Proof: Suppose that f is semi-open let A be any subset of X . Then $f(X-\bar{A})$ is semi-open subset of Y . Therefore, by Theorem 1.1., we have $f(X-\bar{A}) \subset \overline{\text{int}f(X-\bar{A})}$. From this by taking complements, we obtain.

$$\text{int}(\overline{(Y-f(X-\bar{A}))}) \subset Y-f(X-\bar{A})$$

Since f is bijective, then $f(X-\bar{A}) = Y-f(\bar{A})$, consequently we have $\text{int}(\overline{f(\bar{A})}) \subset f(\bar{A})$. This implies that $\text{int}(\overline{f(A)}) \subset f(\bar{A})$.

Corollary 1.3. Let $f: X \rightarrow Y$ be semi-open bijection mapping. Then $f^{-1}(\text{int}\bar{B}) = \overline{f^{-1}(B)}$ for each subset B of Y .

Proof: Let B any subset of Y . Let us put $A = f^{-1}(B)$. Since f is surjective, then $f(A) = B$. Hence, we have $\text{int}\bar{B} \subset f(\overline{f^{-1}(B)})$ by Theorem 1.2. Since f is injective then we obtain $f^{-1}(\text{int}\bar{B}) \subset \overline{f^{-1}(B)}$.

Definition 1.3. (i) A mapping $f: X \rightarrow Y$ is called almost-continuous [5] if for every open V of Y , $f^{-1}(V) \neq \emptyset$ implies $f^{-1}(V) \subset \text{int} f^{-1}(V)$.

(ii) A mapping $f: X \rightarrow Y$ is called feebly-continuous [5] (resp, nearly feebly-continuous [16]) if for every nonempty open set V in Y , $f^{-1}(V) \neq \emptyset$ implies $\text{int} f^{-1}(V) \neq \emptyset$ (resp, $\text{int}(\overline{f^{-1}(V)}) \neq \emptyset$).

(iii) A mapping $f: X \rightarrow Y$ is called feebly-open [5] (resp., nearly feebly-open [16]) if for every nonempty open set U in X , the set $\text{int} f(U)$ (resp., $\text{int}(\overline{f(U)})$) is nonempty.

Remark 1.2. (i) Here almost continuity in Definition 1.3 is precisely semi-continuity in Definition 1.2 by Theorem 1.1.

Furthermore the following implications hold:

(ii) semi-continuous \rightarrow feebly-continuous \rightarrow nearly feebly-continuous

(iii) semi-open \rightarrow feebly-open \rightarrow nearly feebly-open.

These are immediate consequence of their definitions. But none of these implications is reversible as is shown by the following examples.

Example 1.2. Let X be the set of real numbers and \mathcal{U} be the usual topology on X . Let the topology \mathcal{F} on X be generated by $\mathcal{U} \cup \{U \cap Q \mid U \in \mathcal{U}\}$, where Q is the set of rational numbers (see [6, p. 45]). Then the identity mapping $i: (X, \mathcal{U}) \rightarrow (X, \mathcal{F})$ is nearly feebly-continuous, but neither feebly-continuous nor semi-continuous and the inverse of the identity mapping i^{-1} is continuous and nearly feebly-open but it is not feebly-open and semi-open.

Example 1.3. Let X and Y be the set of real numbers with usual topology. Let the mapping $f: X \rightarrow Y$ be defined as follows $f(x) = x$, if $x \neq 0$ and $x \neq 1$; $f(0) = 1$, $f(1) = 0$. Then f is one-to-one feebly-continuous, feebly-open (see [21, p. 174]), but neither semi-continuous nor semi-open.

We give the following result which will be used in sequel.

Theorem 1.4. Let $f: X \rightarrow Y$ nearly feebly-continuous surjection mapping. Then the image of each everywhere dense open subset of X is everywhere dense in Y .

Proof: Let A be any everywhere dense open subset of X . Suppose that $f(A)$ is not everywhere dense subset of Y . Then there is a nonempty open subset G of Y such that $G \cap f(A) = \emptyset$. Consequently we have $f^{-1}(G) \cap A = \emptyset$. From this $\text{int}(\overline{f^{-1}(G)}) \subset X - A$. Since f is nearly feebly-continuous surjection, then $\text{int}(\overline{f^{-1}(G)}) \neq \emptyset$. Let us put $U = \text{int}(\overline{f^{-1}(G)})$. Thus $U \subset X - A$. This shows that $U \cap A = \emptyset$. This is contradiction.

We recall the following Theorem and definitions which will be used in proof of the next theorem.

Theorem 1.5. ([3]). Let A be a subset of X . Then, the following properties are equivalent.

- (i) A is semi-closed
- (ii) $\text{int}\bar{A} \subset A$.
- (iii) $X - A$ is semi-open.

Definition 1.4 ([12]). A mapping $f: X \rightarrow Y$ is said to be semi-closed, if the image $f(F)$ of each closed set F in X is semi-closed in Y .

Remark 1.3. Every closed mapping is semi-closed but the converse is not in general. Indeed the identity mapping which is defined in example of [12, p. 412] is semi-closed, but it is not closed.

Definition 1.5. ([1]) A mapping $f: X \rightarrow Y$ is irreducible if no proper closed subset of X is mapped onto Y by f , or equivalently if any non-empty open subset of X entirely contains some fibre of f , i.e., the set $f^{-1}(y)$ for some $y \in Y$.

As a slight improvement of [6, part (i) of Theorem 4.10, p. 47], we have the following result.

Theorem 1.6. If $f: X \rightarrow Y$ is an irreducible semi-closed mapping. Then f is feebly open.

Proof: Let U be nonempty open subset of X . We put $V = Y - f(X - U)$. Since f is irreducible, then V is nonempty. On the other hand, $V \subset f(U)$. Indeed let y be any point of V . Hence $f^{-1}(y) \subset f^{-1}(V) = f^{-1}(Y - f(X - U))$. From this we have $f^{-1}(y) \subset X - f^{-1}(f(X - U)) \subset X - (X - U) = U$. This shows that $y \in f(U)$. In order to prove that f is feebly-open, it is sufficient to show that $\text{int } V \neq \emptyset$. Since f is semi-closed, then V is semi-open subset of Y by Definition 1.4 and Theorem 1.5. Therefore, by Definition 1.1., there exists a nonempty open subset O of Y such that $O \subset V \subset \overline{O}$. This implies that $\text{int } V \neq \emptyset$. Since $V \subset f(U)$, then f is feebly open.

Definition 1.6. (i) A mapping $f: X \rightarrow Y$ is called almost-open in the sense of Wilansky [22], if for every point $x \in X$ and every neighbourhood U of x , $\overline{f(U)}$ is a neighbourhood of $f(x)$.

(ii) A mapping $f: X \rightarrow Y$ is almost-open in the sense of Singal and Singal [18], if the image of every regular open subset of X is an open subset of Y .

These two kinds of almost-openness of a mapping will be indicated by W -almost-open and S -almost-open respectively in sequel.

Remark 1.4. The following examples show that neither of these two almost open mappings, in general, imply the other.

Example 1.4. Let R be the set of real numbers with usual topology and the topology on $X = \{0,1,2\}$ be generated by $\mathcal{T} = \{X, \emptyset, \{0\}, \{1, 2\}\}$. Let the mapping $f: R \rightarrow X$ be defined as follows: $f(x) = 0$, if x is rational and $f(x) = 1$, if x is irrational. Then f is W -almost-open, but f is not S -almost-open. To see that f is not S -almost-open, while

$U = (0, 1)$ is regularly open subset in $(\mathbb{R}, \mathcal{U})$ but $f(U) = \{0, 1\}$ is not open subset in (X, \mathcal{F}) .

Example 1.5. Let \mathbb{R} be the set of real numbers and \mathcal{F} be the countable complement topology for \mathbb{R} . Let $X = \{a, b\}$ and X be generated by $\mathcal{F}' = \{X, \emptyset, \{a\}\}$. Let the mapping $f: (\mathbb{R}, \mathcal{F}) \rightarrow (X, \mathcal{F}')$ be defined as follows: $f(x) = a$ if x is rational and $f(x) = b$ if x is irrational. Since the regular open subsets of $(\mathbb{R}, \mathcal{F})$ are only \emptyset and \mathbb{R} , then f is S -almost-open, but f is not W -almost-open at any irrational point.

Remark 1.5. (i) Example 1.2 shows that an W -almost-open mapping need not be feebly open. In fact, the inverse of the identity mapping which is defined in example 1.2 is W -almost-open, but it is not feebly-open.

(ii) On the other hand, Example 1.3 shows that a feebly-open mapping is neither S -almost, nor W -almost open indeed, while $G = (-\frac{1}{2}, \frac{1}{2})$ is both regular open and a neighbourhood of 0, consequently $f(G) = (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \cup \{1\}$ is not open and $\overline{f(G)} = [-\frac{1}{2}, \frac{1}{2}] \cup \{1\}$ is not a neighbourhood of $f(0)$, consequently f is neither S -almost open nor W -almost open at 0.

(iii) Furthermore, an S -almost-open mapping need not be feebly-open in general. For example, the mapping of which is defined in Example 1.5 is S -almost open, but while the set $(\mathbb{R}-Q)$, that is, the set of irrational numbers, is open in $(\mathbb{R}, \mathcal{F})$, where Q is the set of rational numbers. Since $f(\mathbb{R}-Q) = \{b\}$ and $\text{int } f(\mathbb{R}-Q) = \emptyset$, then f is not feebly open.

Remark 1.6. Clearly every W -almost-open mapping is nearly feebly-open, but a nearly feebly-open mapping need not be W -almost-open in general. For example, the mapping f which is defined in Example 1.3 is feebly-open. Consequently f is nearly feebly-open because of the part (iii) of Remark 1.2, but f is not W -almost-open by the part (ii) of Remark 1.5.

We recall the following definition about semi-regular spaces which will be needed for the proof of the next theorem.

Definition 1.7. ([20]). A space is said to be semi-regular if for each point of the space and each open set U containing x , there is an open set V such that $x \in V \subset \text{int } \overline{V} \subset U$.

We would like to remark that we have proved the following theorem which was given by T. Noiri in [14] by using the different method.

Theorem 1.7. *If X is a semi-regular and $f: X \rightarrow Y$ is an S -almost-open mapping. Then f is open.*

Proof: Let U be any nonempty open subset of X and let y be any point of $f(U)$, then there is a point x of U such that $y = f(x)$. Since x is semi-regular space, then there exists an open neighbourhood V of x such that $x \in V \subset \text{int } \overline{V} \subset U$. Hence, we have $f(x) \in f(V) \subset f(\text{int } \overline{V}) \subset f(U)$. Since f is S -almost-open, then $f(\text{int } \overline{V})$ is open subset containing $f(x)$. Consequently $f(U)$ is also neighbourhood $f(x)$. This shows that $f(U)$ is open.

The following characterization can be found in [13].

Theorem 1.8. *Let $f: X \rightarrow Y$ be mapping. Then the following are equivalent.*

- (i) f is semi-continuous.
- (ii) $\text{int } (\overline{f^{-1}(B)}) \subset f^{-1}(\overline{B})$ for each subset B of Y .
- (iii) $f(\text{int } \overline{A}) \subset \overline{f(A)}$ for each subset A of X .

The following result shows a relationship between almost-open mappings in two senses.

Theorem 1.9. *Let $f: X \rightarrow Y$ be semi-continuous and S -almost open. Then f is W -almost open.*

Proof: Let $x \in X$ and let U be any neighbourhood of x . Since f is semi-continuous, then $f(\text{int } \overline{U}) \subset \overline{f(U)}$ by Theorem 1.8. On the other hand, since f is S -almost open, then $f(\text{int } \overline{U})$ is open subset containing $f(x)$. Consequently $\overline{f(U)}$ is a neighbourhood of $f(x)$.

Definition 1.8. A mapping $f: X \rightarrow Y$ is said to be weakly-open [17] if $f(U) \subset \text{int } (f(\overline{U}))$ for every open set U of X .

Theorem 1.10. *Let $f: X \rightarrow Y$ be a mapping. Then the following are equivalent.*

- (i) f is weakly-open.
- (ii) for every point $x \in X$ and every neighbourhood V of x , $f(\overline{V})$ is a neighbourhood of $f(x)$.

Proof: (i) \rightarrow (ii): Suppose that f is weakly-open and let x be any point of X and let V be an arbitrary neighbourhood of x . Then there exists an open subset G of X containing x such that $x \in G \subset V$. Hence we have $\text{int } f(\overline{G}) \subset f(\overline{V})$. Since f is weakly-open then $f(G) \subset \text{int } f(\overline{G})$. Thus $f(x) \in f(G) \subset \text{int } f(\overline{G}) \subset f(\overline{V})$. This shows that $f(\overline{V})$ is a neighbourhood of $f(x)$,

(ii) \rightarrow (i): Suppose that (ii) holds, and let G be any nonempty open subset of X and let y be arbitrary point of $f(G)$. Then there exist a point x of G such that $y = f(x)$. Since G is open, then G is a neighbourhood of x . Therefore, by hypothesis, $f(\overline{G})$ is a neighbourhood of $f(x)$. Hence there is a nonempty open subset W of Y containing $f(x) \in W \subset f(\overline{G})$. This implies that $y = f(x) \in \text{int } f(\overline{G})$. Thus $f(G) \subset \text{int } f(\overline{G})$. This completes the proof.

Remark 1.7. (i) Obviously every open mapping is a weakly-open, but the converse is not necessarily true as is shown by the following example.

Example 1.6. Let $X = \{a, b, c, d\}$, $\mathcal{J} = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}, \{b, c, d\}\}$ and $Y = \{1, 2, 3\}$, $\mathcal{J}' = \{Y, \emptyset, \{1\}, \{1, 2\}, \{1, 3\}\}$. Let $f: (X, \mathcal{J}) \rightarrow (Y, \mathcal{J}')$ be given by $f(a) = 1$, $f(b) = 2$, $f(c) = f(d) = 3$. Then f is weakly-open, but f is not open.

(ii) Furthermore, since $f(\overline{\{d\}}) = \{3\}$. This shows that a weakly-open mapping need not be W -almost open in general. On the other hand an W -almost-open mapping may fail to be weakly-open. Indeed, the mapping f which is defined in Example 1.4 is W -almost-open, but while $U = (1, 2)$ is a neighbourhood of $\sqrt{2}$, $f(\overline{U}) = \{0, 1\}$ is not a neighbourhood of $f(\sqrt{2}) = 1$. Hence f is not weakly-open.

(iii) Obviously every S -almost-open mapping is weakly-open, but the converse of this statement may not be true in general. As is shown in Example 1.6, while $\{b\}$ is regular open in (X, \mathcal{J}) , $f(\{b\}) = \{2\}$ is not open in (Y, \mathcal{J}') .

However we do have the following result.

Theorem 1.11. If $f: X \rightarrow Y$ is continuous and weakly-open. Then f is W -almost open.

Proof: Let $x \in X$ and let U be any neighbourhood of x . Since f is continuous, then $f(\overline{U}) \subset \overline{f(U)}$ since f is weakly-open, then $f(\overline{U})$ is

neighbourhood of $f(x)$. Consequently $\overline{f(U)}$ is also neighbourhood of $f(x)$. This shows that f is almost W -open.

We would like to remark that we have the following theorem which was given by D.A. Rose in [17] by using the different method.

Theorem 1.12. If X is a regular space and $f: X \rightarrow Y$ is weakly-open. Then f is an open mapping.

Proof: Let U be any nonempty open subset of X and let y be an arbitrary point of $f(U)$. Then there is a point x of U such that $y = f(x)$. Since X is a regular space, then there exists an open subset V of X containing x such that $x \in \overline{V} \subset U$. From this, $f(x) \in f(\overline{V}) \subset f(U)$. Since f is weakly-open, then $f(\overline{V})$ is a neighbourhood of (x) by Theorem 1.10. This completes the proof of Theorem 1.12.

Before stating the next theorem, we recall the following definitions about weakly-regular space, almost regular space.

Definitions 1.9. A topological space X is said to be weakly-regular (resp, almost regular) [19], if for each point $x \in X$ and each regular open set G containing the closure of singleton $\{x\}$, there exists an open set V (resp, regular open set U) such that $x \in V \subset \overline{V} \subset G$ (resp, $x \in U \subset \overline{U} \subset G$).

Remark 1.8. (i) Clearly every almost-regular space is weakly-regular. But in general a weakly-regular space may fail to be almost regular (e.g., see, [19, Example 2.1, p. 90]).

(ii) Every weakly-regular T_1 -space is almost-regular (see [19, Remark 2.1., p. 90]).

Theorem 1.13. If X is almost-regular and $f: X \rightarrow Y$ is weakly-open, then f is S -almost open.

Proof: Let G be any regular open subset of X and let y be an arbitrary point of $f(G)$. Then there exists a point x of G such that $f(x) = y$. Since X is almost-regular, then there exists a regular open subset V of X containing x such that $x \in V \subset \overline{V} \subset G$. Hence $f(x) \in f(\overline{V}) \subset f(G)$. Since f is weakly-open, then $f(\overline{V})$ is a neighbourhood of $f(x)$ by Theorem 1.10. This shows that $f(G)$ is open in Y . Consequently f is S -almost open.

Corollary 1.14. If X is weakly-regular T_1 -space and $f: X \rightarrow Y$ is weakly-open, then f is S -almost open.

The proof follows from Theorem 1.13 and the part (ii) of Remark 1.8.

Corollary 1.15. If X is almost-regular and $f: X \rightarrow Y$ is semi-continuous and weakly-open, then f is W -almost-open.

The proof follows from Theorem 1.13 and Theorem 1.9.

Corollary 1.16. If X is weakly-regular T_1 -space and $f: X \rightarrow Y$ is semi-continuous and weakly-open, then f is W -almost-open.

The proof is an immediate consequence of the part (ii) of Remark 1.8 and Corollary 1.15.

Definition 1.10. A mapping $f: X \rightarrow Y$ is said to be almost-continuous in the sense of Singal and Singal [18] (resp., in the sense of Husain [7]), if for each point $x \in X$ and for each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset \text{int } \overline{V}$ (resp., $\overline{f^{-1}(V)}$ is a neighbourhood of x).

These two kinds of almost-continuity will be indicated by S -almost-continuous and H -almost-continuous respectively in sequel.

Remark 1.9. (i) The concepts of H -almost-continuous mapping and S -almost continuous mapping are completely independent each other. (e.g. see Example 1 of [11] and Example 2.1 of [18]).

(ii) Example 1.2 shows that an H -almost-continuous mapping need not be feebly-continuous.

(iii) Example 1.3 shows that a feebly-continuous mapping is neither H -almost-continuous nor S -almost-continuous. In fact, while $V = (-\frac{1}{2}, \frac{1}{2})$ is both regular-open and a neighbourhood of $f(1)$. Since $f^{-1}(V) = (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \cup \{1\}$ and $\overline{f^{-1}(V)} = [-\frac{1}{2}, \frac{1}{2}] \cup \{1\}$, then f is neither S -almost-continuous nor H -almost continuous at point 1.

(iv) On the other hand, an S -almost-continuous mapping need not be feebly-continuous. For example, the mapping f which is defined in Example 1.5 is S -almost-continuous, but while the set $\{a\}$ is open in (X, \mathcal{T}') . Since $f^{-1}(\{a\}) = Q$, where Q is rational unumbers set and the interior of Q (with respect to countable complement topology on R) is empty. Consequently f is not feebly-continuous.

(v) Obviously every H -almost-continuous mapping is nearly feebly-continuous. But the converse of this statement is not necessarily true in general. For example, the mapping f which is defined in Example 1.3 is feebly-continuous, consequently f is also nearly feebly-continuous

because of the part (ii) of Remark 1.2. But f is not H -almost-continuous by (iii) of this remark.

P.E. Long and D.A. Carnahan [10] showed that if $f: X \rightarrow Y$ is an S -almost-continuous and open mapping then f is H -almost-continuous. We give the following result which is slightly improved this theorem.

Theorem 1.17. Let $f: X \rightarrow Y$ be an S -almost-continuous and semi-open bijection mapping. Then f is H -almost-continuous.

Proof: Let $x \in X$ and $V \subset Y$ be any neighbourhood of $f(x)$. Since f is S -almost-continuous, then there exists a neighbourhood U of x such that $f(U) \subset \text{int } \bar{V}$. From this we have $U \subset f^{-1}(\text{int } \bar{V})$. Since f is semi-open bijection, then $f^{-1}(\text{int } \bar{V}) \subset \overline{f^{-1}(V)}$ by Corollary 1.3. Consequently, we have $U \subset f^{-1}(V)$. This shows that f is also H -almost continuous.

Definition 1.11. A mapping $f: X \rightarrow Y$ is said to be weakly-continuous [8] (resp., θ -continuous [4]) if for each point $x \in X$ and each neighbourhood V of $f(x)$, there exists a neighbourhood U of x such that $f(U) \subset \bar{V}$ (resp., $f(\bar{U}) \subset \bar{V}$).

Remark 1.10. (i) Obviously every S -almost-continuous mapping is weakly-continuous, but the converse of this statement is not true in general. In fact, the mapping f which is given in Example 2.3 of [18] is weakly-continuous, but it is neither S -almost-continuous nor H -almost-continuous at any rational point.

(ii) It is clear that every θ -continuous mapping is weakly-continuous.

The following results which show that the relationship between weak-continuity and these two forms of almost-continuity, will be used in sequel

Theorem 1.18. If Y is almost-regular and $f: X \rightarrow Y$ is weakly-continuous, then f is S -almost-continuous. [15. Theorem 1. p. 649]

Corollary 1.19. If Y is weakly-regular T_1 -space and $f: X \rightarrow Y$ is weakly-continuous, then f is S -almost-continuous.

The proof follows from the part (ii) of Remark 1.8 and Theorem 1.18.

Corollary 1.20. If Y is almost-regular and $f: X \rightarrow Y$ is weakly-continuous and semi-open bijection mapping, then f is H -almost-continuous.

The proof follows from Theorem 1.18 and Theorem 1.17.

Corollary 1.21. If Y is weakly-regular T_1 -space and $f: X \rightarrow Y$ is weakly-continuous and semi-open bijection mapping, then f is H -almost-continuous.

The proof is an immediate consequence of the part (ii) of Remark 1.8 and Corollary 1.20.

Theorem 1.22. Let $f: X \rightarrow Y$ be weakly-continuous and open mapping. Then f is H -almost-continuous.

Proof: Let $x \in X$ and $V \subset Y$ be any neighbourhood of $f(x)$. Since f is open, then $f^{-1}(\overline{V}) \subset \overline{f^{-1}(V)}$ (see, [10, Lemma, p. 416]). On the other hand, since f is weakly-continuous, then there is an open neighbourhood U of x such that $f(U) \subset \overline{V}$. From this, we have $U \subset f^{-1}(\overline{V}) \subset \overline{f^{-1}(V)}$. This shows that f is H -almost continuous.

We shall require the following results concerning S -almost-continuity, weak-continuity.

Theorem 1.23. (Singal [18]): If $f: X \rightarrow Y$ is a weakly-continuous open mapping, then f is S -almost continuous.

Theorem 1.24. (Singal [18]): If $f: X \rightarrow Y$ is an S -almost-continuous and Y is semi-regular, then f is continuous.

Theorem 1.25. (Levine [8]): If $f: X \rightarrow Y$ is weakly-continuous and Y is regular, then f is continuous.

2. IMAGES OF BAIRE SPACES

We are now in a position to give our main Theorem which is an extension of [6, Proposition 4.4].

Definition 2.1. A mapping $f: X \rightarrow Y$ is called δ -open [6] if for every nowhere dense subset N of Y , $f^{-1}(N)$ is a nowhere dense subset of X , or equivalently for every somewhere dense subset A of X , $f(A)$ is somewhere dense subset of Y .

Theorem 2.1. Let f be a mapping from X into Y . If any one of the following conditions holds, then f is δ -open.

- (i) f is semi-continuous and nearly-feebly open.
- (ii) f is semi-continuous and feebly-open.
- (iii) f is semi-continuous and semi-open.
- (iv) f is semi-continuous and W -almost-open.
- (v) f is semi-continuous and S -almost-open.
- (vi) f semi-continuous and irreducible semi-closed.

Proof: Suppose that (i) holds and let A be somewhere dense subset of X . That is, $\text{int } \bar{A} \neq \emptyset$. Since f is semi-continuous, then $f(\text{int } \bar{A}) \subset \overline{f(A)}$ by Theorem 1.8. Hence we have, $\text{int } (f(\text{int } \bar{A})) \subset \text{int } (\overline{f(A)})$. Since f is nearly feebly open. Then $\text{int } (\overline{f(\text{int } \bar{A})}) \neq \emptyset$, by part (iii) of Definition 1.3. Thus $\text{int } (\overline{f(A)}) \neq \emptyset$. This shows that f is δ -open. Since for each cases of (ii), (iii), (iv) implies the case (i) by means of the part (iii) of Remark 1.2 and Remark 1.6 then for each of these cases, we have also that f is δ -open. Furthermore, we have already seen that the case (v) implies the case (iv) by Theorem 1.9. Hence, we have that if the case (v) holds, f is δ -open. Finally, the proof of the case (vi) follows from the case (ii) by Theorem 1.6.

Theorem 2.2. Let f be a mapping from a Baire space X onto a space Y . If any one of the following conditions holds, then Y is a Baire space.

- (i) f is semi-continuous and nearly feebly-open.
- (ii) f is semi-continuous and W -almost-open.
- (iii) f is semi-continuous and S -almost-open.
- (iv) f is semi-continuous and semi-open.
- (v) f semi-continuous and irreducible semi-closed.

Proof: Suppose that any one of the conditions above mentioned holds and Y is not Baire space. Then there exists nonempty open first category subset B of Y . That is, $B = \bigcup_{n=2}^{\infty} B_n$, where each B_n is nowhere dense subset of Y . Since f is δ -open by Theorem 2.1. then $f^{-1}(B) = \bigcup_{n=2}^{\infty} f^{-1}(B_n)$ is of the first category subset of X . On the other hand since f is semi-continuous then there exists nonempty open subset O of X such that $O \subset f^{-1}(B) \subset \bar{O}$. Thus O is open first category set in X . This contradicts X being Baire space.

It should be noted that a result similar to Theorem 2.2 holds for the space of second category.

Corollary 2.3. Let f be semi-continuous and weakly-open mapping from an almost-regular space X onto a space Y . If X is a Baire space, then Y is also Baire space.

The proof follows from Corollary 1.15 and the case (ii) of Theorem 2.2

Corollary 2.4. Let f be semi-continuous and weakly-open mapping from a weakly-regular T_1 -space X onto a space Y . If X is a Baire space, then Y is also Baire space.

The proof is an immediate consequence of Corollary 1.16 and the case (ii) of Theorem 2.2.

Corollary 2.5. Let f be continuous and weakly-open mapping from a Baire space X onto a space Y . Then Y is also Baire space.

The proof follows from Theorem 1.11 and the case (ii) of Theorem 2.2.

Corollary 2.6. Let f be weakly-continuous and weakly-open from a Baire space onto a regular space Y . Then Y is a Baire space.

The proof follows from Theorem 1.25 and Corollary 2.5.

Corollary 2.7. Let f be S -almost-continuous and weakly-open from a Baire space onto a semi-regular space Y . Then Y is a Baire space.

The proof is direct consequence of Theorem 1.24 and Corollary 2.5.

Corollary 2.8. Let f be S -almost-continuous and S -almost-open from a Baire space onto a semi-regular space Y . Then Y is a Baire space.

The proof is a immediate consequence of Theorem 1.24 and the case (iii) of Theorem 2.2.

It would be interesting to know whether the preimage of a Baire space must be a Baire space under semi-continuous nearly feebly-open bijection mapping. In fact, this is not possible even in the case when the mapping f is continuous nearly feebly-open bijection (e.g., see Example 1.2 of section 1). However, we establish the following result.

Theorem 2.9. Let f be one to one mapping from a space X onto a Baire space Y . If any one of the following conditions holds, then X is a Baire space.

- (i) f is semi-open and nearly feebly-continuous.
- (ii) f is semi-open and H -almost-continuous.
- (iii) f is semi-open and S -almost-continuous.
- (iv) f semi-open and semi-continuous.
- (v) f is open and weakly-continuous.
- (vi) f is open and θ -continuous.

The proof of this theorem is mainly based on the following Lemma.

Lemma 2.10. Let f be one-to-one mapping from X onto Y . If any one of the following conditions holds, then the image of each nowhere dense subset of X , is nowhere dense subset of Y .

- (i) f is semi-open and nearly feebly-continuous.
- (ii) f is semi-open and feebly-continuous.
- (iii) f is semi-open and semi-continuous.
- (iv) f is semi-open and H -almost-continuous.
- (v) f is semi-open and S -almost-continuous.
- (vi) f is open and weakly-continuous.
- (vii) f is open and θ -continuous.

Proof: Suppose that (i) holds and \bar{N} nowhere dense subset of X . That is, $\text{int } \bar{N} = \emptyset$. From this we have $\overline{X - \bar{N}} = X$. Since f is nearly feebly-continuous, then $f(\overline{X - \bar{N}}) = Y$ by Theorem 1.4. On the other hand, since f is one-to-one, then $f(X - \bar{N}) \subset Y - f(\bar{N})$. Therefore, we have $\overline{Y - f(\bar{N})} = Y$. Now by taking complements, we obtain $\text{int}f(\bar{N}) = \emptyset$. Since f is semi-open, then $\text{int } (\overline{f(\bar{N})}) \subset f(\bar{N})$ by Theorem 1.2. Hence $\text{int } (\overline{f(\bar{N})}) = \emptyset$. It is clear that each cases of (ii), (iii), (iv) implies the case (i) by the part (ii) of Remark 1.2 and part (v) of Remark 1.9. Therefore, the proof from the case (ii) to the case (iv) follows from the proof of the case (i). We have seen that the case (v) implies the case (iv) by Theorem 1.17. Hence the proof of the case (v) follows from the case (iv). Furthermore, since the case (vi) implies the case(v) by Theorem 1.23,

then the proof of the case (vi) is direct consequence of the proof of case (v). Finally it is clear that the case (vii) implies the case (vi) by part (ii) of Remark 1.10.

Let us give the proof of Theorem 2.9.

Suppose that any one of the conditions which is given in the Theorem 2.9. holds and X is not Baire space. Then there is nonempty open first category subset A of X . That is, $A = \bigcup_{n=2}^{\infty} A_n$, where each A_n is nowhere dense subset in X . By Lemma 2.10, $f(A) = \bigcup_{n=2}^{\infty} f(A_n)$ is of the first category subset of Y . Since f is semi-open, then $f(A)$ is semi-open subset of Y . Therefore, there is nonempty open subset O of Y such that $O \subset f(A) \subset \bar{O}$. Thus O is an open first category subset in Y . This contradicts Y being Baire space.

Corollary 2.11. Let f be one-to-one mapping from a semi-regular space X onto Baire space Y . If any one of the following conditions holds, then X is also Baire space

- (i) f is S -almost-open and weakly-continuous.
- (ii) f is S -almost-open and θ -continuous.

The proof of the case(i) follows from Theorem 1.7 and the case (v) of Theorem 2.9 The proof of the case (ii) is an immediate consequence of the case (i) of this corollary and part (ii) of Remark 1.10.

Corollary 2.12. Let f be one-to-one S -almost-open and S -almost-continuous from a semi-regular space X onto a Baire space Y . Then X is also Baire space.

The proof follows from Theorem 1.7 and the case (iii) of Theorem 2.9.

Corollary 2.13. Let f be one-to-one weakly-open and weakly-continuous mapping from regular space X onto a Baire space Y . Then X is also Baire space.

The proof is an immediate consequence of Theorem 1.12 and the case (v) of Theorem 2.9.

Corollary 2.14. Let f be one-to-one semi-open and weakly-continuous from a space X onto an almost-regular space Y . If Y is a Baire space, then X is a Baire space.

The proof follows from Corollary 1.20 and the case (ii) of Theorem 2.9.

Corollary 2.15. Let f be one to one semi-open and weakly-continuous from a space X onto a weakly-regular T_1 -space Y . If Y is a Baire space, then X is a Baire space.

The proof is direct consequence of Corollary 1.21 and the case (ii) of Theorem 2.9.

Corollary 2.16 Let f be one-to-one semi-open and semi-continuous mapping from X onto Y . Then X is a Baire space if and only if Y is a Baire space.

The proof follows from the case (iv) of Theorem 2.2. and the case (iv) of Theorem 2.9.

Corollary 2.17. Let f be one-to-one semi-open and weakly-continuous mapping from a space X onto a regular space Y . Then X is a Baire space if and only if Y is a Baire space.

The proof follows from Theorem 1.25 and Corollary 2.16.

Corollary 2.18. Let f be one-to-one weakly-open and semi-continuous mapping from a regular-space X onto a space Y . Then X is a Baire space if and only if Y is a Baire space.

The proof is an immediate consequence of Theorem 1.12 and Corollary 2.16.

Corollary 2.19. Let f be one-to-one semi-open and S -almost-continuous mapping from a space X onto a semi-regular space Y . Then X is a Baire space if and only if Y is a Baire space.

The proof follows from Theorem 1.24 and Corollary 2.16.

Corollary 2.20 Let f be one to one S -almost-open and semi-continuous mapping from a semi-regular space X onto a space Y . Then X is a Baire space if and only if Y is a Baire space.

The proof is direct consequence of Theorem 1.7 and Corollary 2.16.

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