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RELATIONS BETWEEN THE SCALAR CURVATURES OF SUBMANIFOLDS WITH CONSTANT CURVATURE

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ABSTRACT

In this paper, the relations between the scalar curvatures of n-dimensional submanifold (hypersurface) N, with zero curvature immersed in an (n+1) –dimensional submanifold $\overline{N}$ with zero curvature in E^m ($m>n+1$), have been investigated and some results have been obtained in terms of scalar, Gaussian and meaan curvtures of the submanifolds N and $\overline{N}$.

INTRODUCTION

We shall assume throughout that all manifolds, maps, vector fields, etc... are differentiable of class C^∞.

Suppose that $\overline{N}$ is an (n+1)-dimensional submanifold of the Euclidean space E^m ($m>n+1$), and N is an n-dimensional hypersurface immersed in an (n+1)-dimensional submanifolds $\overline{N}$ with constant curvature K. Let p, be a point of N and x^i the local coordinates around p in N such that $X_i=\partial_i$ form an orthonormal basis of T_p (N) at the point p, ζ be orthonormal normal vector field of N in $\overline{N}$, X and Y be two linear independent vectors at the point p and γ (X,Y) be the plane section spanned by X and Y. On the other hand, $K(\gamma)$ is the constant for all plane sections γ in the tangent space $T_p(N)$ at p where $p\in N$, then N is a hypersurface with the constant curvature. The standard Riemann connection of E^m and Riemann connections of $\overline{N}$ and N are denoted by

$\bar{\bar{D}}$, $\bar{D}$ and D, respectively.

The Weingarten map L of N in $\overline{N}$ is given by

$$\bar{D}_X\zeta = L(X), \; A \; X \in N_p \tag{1.1}$$

and det L is the Gauss curvature at the point p of the hypersurface N of $\overline{N}$.

Definition 1.1. Let M be an n-dimensional submanifold of the Euclidean space E^m. Then

$\alpha: \chi(M) \times \chi(M) \rightarrow \chi(M)^{\perp}$

$$(Y,Z) \rightarrow \alpha(Y,Z) = \sum_{j=1}^{m-n} \alpha^j(Y,Z)\zeta_j \qquad (1.2)$$

is called second fundamental form of M. Where α^j denotes the coefficients of the second fundamental vector field in the direction of ζ_j, that is,

$$\alpha^j(Y,Z) = < \alpha(Y,Z), \zeta_j >. \qquad [1]$$

To be Y, Z $\in \chi(N)$, Let $\alpha_1(Y,Z)$ be the second fundamental form of $\tilde{N}$ in E^m, then we have

$$\bar{\bar{D}}_Y Z = \tilde{D}_Y Z + \alpha_1(Y,Z) \qquad (1.3)$$

and if $\alpha_1(Y,Z)$ is the second fundamental form of N in E^m, then we have

$$\bar{\bar{D}}_Y Z = D_Y Z + \alpha_2(Y,Z). \qquad (1.4)$$

If Y and Z are vector fields of N, then we have

$$\tilde{D}_Y Z = D_Y Z + \alpha_3(Y,Z). \qquad (1.5)$$

Here (1.5) is the Gauss equation of N in $\tilde{N}$, where α_3 (Y,Z) is the second fundamental form N in $\tilde{N}$.

If we consider (1.5) and

$$\alpha_3(Y,Z) = - < L(Y,)Z > \zeta \qquad (1.6)$$

we obtain

$$\tilde{D}_Y Z = D_Y Z - < L(Y),Z > \zeta, \qquad (1.7)$$

and using (1.7) in (1.3) we have

$$\bar{\bar{D}}_Y Z = D_Y Z - < L(Y),Z > \zeta + \alpha_1(Y,Z). \qquad (1.8)$$

Moreover, if we consider (1.4) and (1.8) then we have

$$\alpha_2(Y,Z) = - < L(Y),Z > \zeta + \alpha_1(Y,Z). \qquad (1.9)$$

Let X and Y be orthonormal vectors at a point p and γ (X,Y) be the plane spanned by X and Y. The sectional curvature $K(\gamma)$ for γ (X,Y) is defined by

$$K(\gamma) = K(X,Y,X,Y)$$

or

$$K(\gamma) = < X,R(X,Y)\ Y >$$

where R is the curvature tensor.

It is easy to see that $K(\gamma)$ is independent of the choice of an orthonormal basis. So, we may give the following definition.

Definition 1.2. If $K(\gamma)$, is a constant for all plane in the tangent space $T_p(M)$ at p for all points $P \in M$, then M is called a space of constant curvature [2].

Let M be an n-dimensional manifold immersed in an m-dimensional Riemann manifold N of constant curvature K, p be a point of M and X^i the local coordinates around p in M such that $X_i = \partial_i$ form an orthonormal basis of $T_p(M)$ at p and also ζ_X be the orthonormal normal vector field of M. If we substitute

$$\alpha(X_i,X_j) = \alpha^X(X_i,X_j)\zeta_X = a^x{}_{ij}\zeta_X$$

then, we have $\alpha^x{}_{ji} = \alpha^x{}_{ij}$. Let $< \alpha >$ denote the length of the second fundamental form α, that is

$$< \alpha, \alpha > = < \alpha >^2 = \alpha^x{}_{ji}\, \alpha_x{}^{ji},$$

where $\alpha_X{}^{ji} = g^{jt} g^{is} \alpha^x{}_{ts}$.

Definition 1.3. If E_1, $E_1, \ldots,$ E_n are local orthonormal vector fields, then

$$R(X,Y) = \sum_{i=1}^{n} g\,(K(E_i,X)\,Y,E_i)$$

$$= \sum_{i=1}^{n} k\,(E_i,Y,E_i,X)$$

defines a global tensör field R of type (0,2) with local components

$$K_{ji} = K_{tji}{}^t = g^{ts}K_{tjis}.$$

Moreover, from the tensor field R we can define a global scalar field

$$r = \sum_{i=1}^{n} R(E_i,E_i)$$

with local components

$$r = g^{ij}K_{ji}.$$

The tensor field R and the function r are called the Ricci tensor and scalar curvature.

From the Gauss equation, we find that the scalar curvature r and the mean curvature vector H satisfy the following relation.

$$r = n^2 \|H\|^2 - < \alpha >^2 + n\,(n-1)\,K. \ [2]$$

Theorem 1.1. Let r be the scalar curvature of n-dimensional submanifold N with zero curvature and $\bar{r}$ be the scalar curvature of (n+1) - dimensional submanifold $\overline{N}$ with zero curvature in E^m. Then, the relation between the scalar curvature of N and the scalar curvature of $\overline{N}$ is given by

$$\bar{r} - r = (n+1)^2 \|\overline{H}\|^2 - n^2 \|H\|^2 - 2 \sum_{i=1}^{n} < \alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta) >$$

$$- < \alpha_1(\zeta,\zeta),\alpha_1(\zeta,\zeta) > - (H^0)^2,$$

in E^m, where $(H^0)^2 = \sum_{i=1}^{n} \lambda_i^2$ and $\lambda_i = < L(e_i),e_i >$.

Proof: By the hypothesis, we have

$$S_p \{e_1,e_2,\ldots,e_n,e_{n+1} = \zeta\} = \chi(N)$$

and

$$S_p \{e_1,e_2,\ldots,e_n\} = \chi(N).$$

Furthermore, since K=0 for the scalar curvature of M at the point p∈M, by hypothesis from the following equation

$$r = n^2 \|H\|^2 - < \alpha >^2 + n\,(n-1)\,K,$$

we have

$$r = n^2 \|H\|^2 - < \alpha >^2. \qquad (1.10)$$

If we consider (1.9), we have

$$\alpha_2(e_i,e_j) = - < L(e_i),e_j > \zeta + \alpha_1(e_i,e_j).$$

From (1.2), it follows that

$$< \alpha_2 >^2 = \sum_{i,j=1}^{n} < \alpha_2(e_i,e_j),\ \alpha_2(e_i,e_j) >.$$

Thus,

$$<\alpha_2>^2 = \sum_{i,j=1}^{n} <\alpha_1(e_i,e_j),\alpha_1(e_i,e_j)> + \sum_{i=1}^{n} \lambda_i^2, \text{ where } \lambda_i = < L(e_i), e_i >. \qquad (1.11)$$

In the same way, from (1.2), we have

$$< \alpha_1 >^2 = \sum_{i,j=1}^{n+1} < \alpha_1(e_i,e_j), \alpha_1(e_i,e_j) >$$

or

$$< \alpha_1 >^2 = \sum_{i,j=1}^{n} <\alpha_1(e_i,e_j), \alpha_1(e_i,e_j) >$$

$$+ 2 \sum_{i=1}^{n} <\alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta)> + <\alpha_1(\zeta,\zeta),\alpha_1(\zeta,\zeta)> \qquad (1.12)$$

since N and $\overline{N}$ are manifolds with zero curvature in E^m and using the equation (1.10), (1.11) and (1.12) we obtain

$$\bar{r} - r = (n+1)^2 \|\overline{H}\|^2 - n^2 \|H\|^2 - 2 \sum_{i=1}^{n} <\alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta) >$$

$$- < \alpha_1(\zeta,\zeta), \alpha_1(\zeta,\zeta) > - (H^0)^2. \qquad (1.13)$$

This completes the proof.

Corollary 1.1. If the scalar curvature of N is zero and if ζ is asymptotic in $\overline{N}$, then

$$\bar{r} = (n+1)^2 \|\overline{H}\|^2 - 2 \sum_{i=1}^{n} < \alpha_1 (e_i,\zeta),\alpha_1(e_i,\zeta)> - (H^0)^2.$$

Proof: Since the scalar curvature of N is zero and ζ is asymptotic in $\overline{N}$ the proof is trivial by (1.10) and (1.13).

Corollary 1.2. If the scalar curvature of $\overline{N}$ is zero, then

$$r = n^2 \|H\|^2 + 2 \sum_{i=1}^{n} < \alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta) > + (H^0)^2.$$

Proof: Since the scalar curvature of $\overline{N}$ is zero, the proof is trivial by (1.10) and (1.13).

Corollary 1.3. Let $p \in \overline{N}$. If $(e_i)_p$ and ζ_p are conjugate two tangent vectors and if ζ_p is asymptotic, then

$$\bar{r} - r = (n+1)^2 \|\overline{H}\|^2 - n^2 \|H\|^2 - (H^0)^2.$$

Proof: Since, $(e_i)_p$ and ζ_p are conjugate and ζ_p is asymptotic, then the requirement results is obtained.

From definition 1.1 we write

$$\alpha_1(e_i,e_i) = \sum_{k=1}^{m-n} \alpha^k(e_i,e_i)\zeta_k.$$

For $\zeta_k \in \mathcal{X}(N)^\perp$ we have

$$< \alpha_2(e_i,e_i), \zeta_k > = \alpha^k (e_i,e_i)$$

or

$$\alpha_2(e_i,e_i) = \sum_{k=1}^{m-n} < \alpha_2(e_i,e_i),\zeta_k > \zeta_k. \qquad (1.14)$$

Denoting the metric connection of the normal bundle $N^\perp$ in E^m by $D^\perp$, we write for $e_i \in \mathcal{X}(N)$

$$\bar{\bar{D}}_{e_i}\zeta_k = - A\ \zeta_k(e_i) + D^\perp{}_{e_i}\zeta_k$$

or

$$< \bar{\bar{D}}_{e_i}\zeta_k,e_i > = < - A\zeta_k(e_i) + D^\perp{}_{e_i}\zeta_k,e_i >.$$

Then we get

$$< \alpha_2(e_i,e_i),\zeta_k > = < A\zeta_k(e_i), e_i >. \qquad (1.15)$$

Thus from (1.14) and (1.15) we have

$$\alpha_2(e_i,e_i) = \sum_{k=1}^{m-n} < A\zeta_k(e_i), e_i > \zeta_k \qquad (1.16)$$

and

$$\alpha_2(e_j,e_j) = \sum_{i=1}^{m-n} \Sigma <A\zeta_1(e_j),e_j > \zeta_1. \qquad (1.17)$$

Using (1.16) and (1.17) we write for $k=1$

$$\sum_{i,j=1}^{n} <\alpha_2(e_i,e_i),\alpha_2(e_j,e_j)> = \sum_{i,j=1}^{n} \sum_{k=1}^{m-n} <A\zeta_k(e_i),e_i><A\zeta_k(e_j),e_j> \qquad (1.18)$$

considering that $\sum_{i=1}^{n} A\zeta_k(e_i) = \sum_{i,j=1}^{n} a_{ij}e_j$ we get

$$\sum_{i=1}^{n} < A\zeta_k(e_i),e_i > = \sum_{i,j=1}^{n} < a_{ij}e_j,e_j >$$

or

$$\sum_{i=1}^{n} < A\zeta_k(e_i),e_i > = \sum_{i=1}^{n} a_{ii},\ _{i=j}.$$

Hence we have obtained that

$$\text{tr } A\zeta_k = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} < A\zeta_k(e_i), e_i > \tag{1.19}$$

or

$$\text{tr } A\zeta_k = \sum_{j=1}^{n} a_{jj} = \sum_{j=1}^{n} < A\zeta_k(e_j), e_j > \tag{1.20}$$

and that

$$\sum_{k=1}^{m-n} (\text{tr } A\zeta_k)^2 = \sum_{i,j=1}^{n} \sum_{k=1}^{m-n} < A\zeta_k(e_i), e_i > < A\zeta_k(e_j), e_j >. \tag{1.21}$$

On the other hand we have

$$\|H\| = \sum_{k=1}^{m-n} (\text{tr } A\zeta_k / n)\zeta_k$$

and so

$$n^2 \|H\|^2 = \sum_{k=1}^{m-n} (\text{tr } A\zeta_k)^2. \tag{1.22}$$

Then from (1.18), (1.21) and (1.22) we get

$$\sum_{i,j=1}^{n} < \alpha_2(e_i,e_i), \alpha_2(e_j,e_j) > = n^2 \|H\|^2.$$

This gives for i=j,

$$H = 1/n \sum_{i=1}^{n} \alpha (e_i,e_i).$$

If H=0 at each point of N then N is minimal and so $\alpha = 0$. From (1.9), we write

$$\alpha_1(e_i,e_i) = < L(e_i), e_i > \zeta.$$

Since the hypersurface N is totaly geodesic, L=O and so $\alpha_1 = 0$. Then from

$\overline{H} = 1/n+1 \sum_{i=1}^{n+1} \alpha_1(e_i,e_i)$ we have that $\overline{H} = 0$, that is the submanifold $\overline{N}$ is minimal and also from $\alpha_1(\zeta,\zeta) = 0$, we can say that ζ is an asymptotic direction in $\overline{N}$. Therefore vwe have proved the assertion.

Application 1.1. Let $\bar{N}_1$ be an 3-dimensional submanifold in E^m, given by the following parametric form

$$X = \{(a+k/\sqrt{2})\ \cos u.\ \cos v,\ (a+k/\sqrt{2})\ \cos u.\ \sin v,\ (a+k/\sqrt{2})\ \sin u,\ k/\sqrt{2}, 0, \ldots, 0) \mid x_j = 0, j = 5, 6, \ldots, m,\ k \in IR\}$$

and let S^2 be a 2–hypersphere in E^m, given by the following parametric form $Y = \{(a.\cos u.\ \cos v,\ a.\ \cos u.\ \sin v,\ a.\ \sin u, 0, \ldots, 0) \mid y_j = 0, j = 4, 5, \ldots, m, a > 0\}$, If the scalar curvature of S^2 and $\bar{N}_1$ are, respectively, r_b and $\bar{r}_a$ in E^m, then

$$\bar{r}_a - r_b = 9\,\|\bar{H}_a\|^2 - 4\,\|H_b\|^2 - \sin u/2(a+k/\sqrt{2})^2 + (H^0)^2.$$

Indeed, we may write

$$y_1 = x_1 = e_1 = (-\sin u \cos v, -\sin u \sin v, \cos u, 0, \ldots, 0)$$

$$y_2 = x_2 = e_2 = (-\sin v, \cos v, 0, \ldots, 0)$$

$$x_3 = \zeta_0 = (1/\sqrt{2} \cos u \cos v, 1/\sqrt{2} \cos u \sin v, 1/\sqrt{2} \sin u, -1/\sqrt{2}, 0, \ldots 0) \tag{1.23}$$

then

$$Sp\ \{e_1|_p, e_2|_p\} = T_{S^2}(p),$$

$$S_p\ \{e_3|_p = \zeta_0|_p, \zeta_1|_p, \partial/\partial x_5|_p, \ldots, \partial/\partial x_m|_p\} = T^{\perp}_{S^2}(p)$$

and

$$Sp\ \{e_1|_p, e_2|_p, e_3|_p = \zeta_0|_p\} = T\bar{N}_1(p),$$

$$Sp\ \{\zeta_1|_p, \partial/\partial x_5|_p, \ldots, \partial/\partial x_m|_p\} = T^{\perp}\bar{N}_1(p).$$

From (1.9) and (1.2) we have

$$\alpha_b(e_i,e_j) = -< L(e_i), e_j > \zeta_0 + \alpha_a(e_i,e_j),$$

$$< \alpha_b >^2 = \sum_{i,j=1}^{2} < \alpha_b(e_i,e_j), \alpha_b(e_i,e_j) > + \sum_{i=1}^{2} \lambda_i^2 \tag{1.24}$$

and

$$< \alpha_a >^2 = \sum_{i,j=1}^{2} < \alpha_a(e_i,e_j), \alpha_a(e_i,e_j) >$$

$$+ 2 \sum_{i=1}^{2} <\alpha_a(e_i,\zeta_0), \alpha_a(e_i,\zeta_0)> + <\alpha_a(\zeta_0,\zeta_0), \alpha_a(\zeta_0,\zeta_0)>. \tag{1.25}$$

Then, from (1.10), (1.24) and (1.25) we obtain

$$\bar{r}_a - r_b = 9\,\|\bar{H}_a\|^2 - 4\,\|\bar{H}_b\|^2 - 2 \sum_{i=1}^{2} < \alpha_a(e_i,\zeta_0),\alpha_a(e_i,\zeta_0) >$$

$$- < \alpha_a(\zeta_0,\zeta_0),\alpha_a(\zeta_0,\zeta_0)> - (H^0)^2. \qquad (1.26)$$

If we put the values of e_1, e_1 and ζ_o, given by (1.23), in (1.26) then we obtain

$$\bar{r}_a - r_b = 9\ \|\bar{H}_a\|^2 - 4\ \|H_b\|^2 - \sin u/2\ (a+k/\sqrt{2})^2 + (H^0)^2.$$

REFERENCES

[1] NOMIZU, K., "Foundation of Differential Geometry" John Willey & Sons, New York, 1969.

[2] CHEN, B.Y., "Geometry of Submanifolds" Marcel Dekker, New York, 1973.