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NORMAL SUBGROUPS OF THE HECKE GROUP H ($\sqrt{2}$)*

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1. INTRODUCTION

Hecke groups H(λ) are the discrete subgroups of PSL(2, **R**) (the group of orientation preserving isometries of the upper half plane U) generated by two linear fractional transformations

$$R(z) = -1/z \text{ and } T(z) = z + \lambda$$

where $\lambda \in \mathbf{R}$, $\lambda \geq 2$ or $\lambda = \lambda_q = 2\cos(\pi/q)$, $q \in N$, $q \geq 3$. These values of λ are the only ones that give discrete groups, by a theorem of E. Hecke. We are going to be interested in the latter case $\lambda = \lambda_q$. The element S = RT is then elliptic of order q.

It is well–known that $H(\lambda_q)$ is the free product of two cyclic groups of orders 2 and q, i.e.

$$H(\lambda_q) \cong C_2 * C_q$$

so that the signature of $H(\lambda_q)$ is (O; 2, q, ∞).

Most important and worked Hecke group is the modular group $\Gamma = H(\lambda_3) = H(1)$. Its underlying field is Q, i.e. all coefficients are rational integers.

Next two important Hecke groups are those for q = 4 and 6. In these cases $\lambda_q = \sqrt{2}$ and $\sqrt{3}$, therefore underlying fields are quadratic extensions of Q by $\sqrt{2}$ dnd $\sqrt{3}$, respectively. Here we only discuss the case q = 4 (q = 6 is similar).

$H(\lambda_4)$ is the only Hecke group, apart from Γ dnd $H(\lambda_\sigma)$, whose elements are completely known. Indeed it consists of the set of all matrices of the following two types:

* This work is produced from the author's PhD Thesis.

$$\text{(i)} \quad \begin{bmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{bmatrix} \quad ; \ ad-2bc = 1,$$

$$\text{(ii)} \quad \begin{bmatrix} a\sqrt{2} & b \\ c & d\sqrt{2} \end{bmatrix} \quad ; \ 2ad-bc = 1.$$

Those of type (i) are called even while the others are called odd.

The set of all even elements form a normal subgroup, $H_e(\sqrt{2})$, of index 2 in $H(\sqrt{2})$, called the even subgroup. It is the free product of the infinite cyclic group Z with a finite cyclic group of order 2. Indeed, being odd elements, R and S both go to 2–cycles under the homomorphism

$$H(\sqrt{2}) \longrightarrow H(\sqrt{2}) / H_e(\sqrt{2}) \cong C_2,$$

i.e.

$$\begin{aligned} R &\longrightarrow (1\ 2) \\ S &\longrightarrow (1\ 2) \\ T &\longrightarrow (1)\,(2), \end{aligned}$$

so by a theorem of D. Singerman [Si], the signature of $H_e(\sqrt{2})$ is $(0;\ 2,\ \infty,\ \infty)$. If we choose I, R as a Schreier transversal for $H(\sqrt{2})/H_e(\sqrt{2})$ then by the Reidemeister–Schreier method, $H_e(\sqrt{2})$ has the parabolic generators T and $U = SR$ with their product TU being the elliptic generator of order 2.

$H_e(\sqrt{2})$ is quite important amongs the normal subgroups of $H(\sqrt{2})$. It is one of the three normal subgroups with cyclic quotient C_2, and contains infinitely many normal subgroups of $H(\sqrt{2})$.

$H(\sqrt{2})$ and $H(\sqrt{3})$ are the anly Hecke groups commensurable with the modular group Γ. Although a conjugate of $H(\sqrt{2})$ and Γ have a common subgroup, no common normal subgroup in both of them exists. To see this let us suppose there exists a normal subgroup N in Γ and $H(\sqrt{2})^M$ where $H(\sqrt{2})^M$ denotes the conjugation by M. Let $\eta(N)$ be the normalizer of N in $PSL(2, \mathbb{R})$. Now $H(\sqrt{2})^M$ contains the element S^M of order 4 and $S^M \in \Gamma$. But $\eta(N)$ contains Γ and S^M. As $\eta(N)$ is also Fuchsian, this contradicts maximality of Γ (see [1]).

Here we are going to discuss some normal subgroups of $H(\sqrt{2})$. They seem to be more numerous than normal subgroups of Γ. Our main concern will be genus 0 and geneus 1 subgroups, congruence subgroups and some relations with the regular maps.

Being a free product of two cyclic groups of orders 2 and 4, by the Kurosh subgroup theorem, $H(\sqrt{2})$ has two kinds of subgroups those which are free and those with torsion (being free product of C_2's, C_4's and Z's).

2. NORMAL SUBGROUPS OF GENUS O IN $H(\sqrt{2})$

Let N be such a subgroup. $H(\sqrt{2})/N$ is a group of automorphisms of $U/N \cong$ Sphere ($U = U \cup Q \cup \{\infty\}$), so it must be isomorphic to a finite subgroup of SO(3), which is going to be a finite triangle group. These are known as $A_5 \cong (2, 3, 5)$, $S_4 \cong (2, 3, 4)$, $A_4 \equiv (2, 3, 3)$, $D_n \cong (2, 2, n)$ and $C_n \cong (1, n, n)$.

Let's first map $H(\sqrt{2})$ onto a cyclic group C_n. Since S must go to n–cycles, n must divide 4. Therefore $n = 1, 2$ or 4. Here N has the signature $(O; 2^{(n)}, 4/n, \infty)$ and therefore is isomorphic to the free product of $C_{4/n}$ and n C_2's. It shall be denoted by $Y_n(\sqrt{2})$.

Secondly, by mapping onto the dihedral group $D_n \cong (2, n, 2)$ we similarly obtain a subgroup with signature $(O; 4/n^{(2)}, \infty^{(n)})$ where $n \mid 4$. We'll denote this one by $S_n(\sqrt{2})$. Note that $S_1(\sqrt{2})$ and $S_2(\sqrt{2})$ contain elements of finite order while $S_2(\sqrt{2})$ is free of rank 3.

Thirdly, if we map onto $S_4 \cong (2, 4, 3)$, we obtain a normal subgroup with signature $(o; \infty^{(8)})$ denoted by $T(\sqrt{2})$. It is isomorphic to a free group of rank 7.

We have already got 7 normal subgroups of genus 0. Apart from these, there is an infinite family of such subgroups, obtained by mapping onto $D_n \cong (2, 2, n)$, $n \in N$. The obtained subgroup has signature $(0; 2^{(n)}, \infty, \infty)$ and will be denoted by $W_n(\sqrt{2})$. Each of these contains infinitely many others of the same kind since $W_n(\sqrt{2}) \triangleright W_{nk}(\sqrt{2})$, $k \in N$. Note that $W_1(\sqrt{2}) = H_e(\sqrt{2})$ and also that $W_2(\sqrt{2}) = S_2(\sqrt{2})$.

Theorem 1. All normal subgroups of genus 0 in $H(\sqrt{2})$ are $H(\sqrt{2})$, $Y_2(\sqrt{2})$, $Y_4(\sqrt{2})$, $S_1(\sqrt{2})$, $S_4(\sqrt{2})$, $T(\sqrt{2})$ and $W_n(\sqrt{2})$ for $n \in N$.

Therefore unlike odd q case (particularly modular group), we have infinitely many normal subgroups of genus 0.

3. FREE NORMAL SUBGROUPS OF $H(\sqrt{2})$

We first have

Lemma 1. The only normal subgroups of $H(\sqrt{2})$ containing elements of finite order are $H(\sqrt{2})$, $Y_2(\sqrt{2})$, $Y_4(\sqrt{2})$, $S_1(\sqrt{2})$ an $W_n(\sqrt{2})$, $n \in N$.

Note that unlike odd q case (particularly modular group), $H(\sqrt{2})$ has infinitely many normal subgroups with elements of finite order. As a result we have.

Corollary 1. Let N be a normal subgroup of positive genus in $H(\sqrt{2})$. Then N is torsion–free.

Corollary 1 does not have a converse, i.e. there are free normal subgroups of $H(\sqrt{2})$ with genus 0.

Theorem 2. Let N be a non–trivial normal subgroup of $H(\sqrt{2})$ different from $H(\sqrt{2})$, $Y_2(\sqrt{2})$, $Y_4(\sqrt{2})$, $S_2(\sqrt{2})$ and $W_n(\sqrt{2})$, $n \in N$. Then N is free.

It is well-known that a free normal subgroup N of $H(\sqrt{2})$ will have rank $r = 2g + t-1$, where t is the parabolic class number of N. Also if $[H(\sqrt{2}): N] = \mu$, then $4 \mid \mu$ as R goes to $\mu / 2$ 2–cycles and S goes to $\mu / 4$ 4–cycles. By the Riemann–Hurwitz formula the genus g of N is

$$g = 1 + \mu \frac{n-4}{8n} .$$

Therefore for $g \neq 1$, $H(\sqrt{2})$ can only have finitely many normal free subgroups of genus g. For $g = 1$, using regular maps of type $\{4, 4\}$, we shall prove that $H(\sqrt{2})$ has infinitely many such subgroups, as the last equation suggests.

4. NORMAL SUBGROUPS OF GENUS 1 IN $H(\sqrt{2})$

Rosenberger and Kern–Isberner have discussed these subgroups in [6]. Here we consider them briefly using their connection with the regular maps.

Let N be a normal subgroup of genus 1 in $H(\sqrt{2})$. We know that N is free of rank $r = t + 1$, of level 4 and therefore of index μ divisible by 4, in $H(\sqrt{2})$. It is shown that each normal subgroup corresponds to a regular map the same genus (see [5]). As N has genus 1, the corresponding regular map M must be of type $\{4, 4\}$ since $S^4 = I$. These are classified as $\{4, 4\}_{r,s}$; $r, s \in N \cup \{O\}$. M has t vertices, 2t edges and t faces where $t = r^2 + s^2$. Each $\{4, 4\}_{r,s}$ will give us a normal subgroup N with index $\mu = 4\,(r^2 + s^2)$ in $H(\sqrt{2})$, since $|\,\mathrm{Aut}\ M\,| = 4\,(r^2 + s^2)$. Hence we have

Theorem 3. $H(\sqrt{2})$ has infinitely many normal subgroups of genus 1.

Now $\mu = 4\,(r^2 + s^2) = 4t$ implying that $t = r^2 + s^2$.

Let μ be given (equivalently t be given). We want to find the number $N_4(\mu)$ of normal subgroups of $H(\sqrt{2})$ with $g = 1$ and index μ.

The number of solutions of $t = r^2 + s^2$ is always divisîble by 4. This is because all the pairs (r, s), (–r, –s), (–r, s) and (r, –s) give the same t. Therefore we have

Theorem 4. The number of normal subgroups of genus 1 with a given index $\mu = 4t$ in $H(\sqrt{2})$ is

$$N_4(\mu) = 1/4.\ \{(r, s) \in Z^2 \mid r^2 + s^2 = t\}.$$

Rosenberger & Kern–Isberner proved this result using the multiplicativity of $N_4(\mu)$. The first few values of $N_4(\mu)$ are as follows:

μ	4	8	12	16	20	24	28	32	36	40
$N_4(\mu)$	1	1	0	1	2	0	0	1	1	2

5. NORMAL SUBGROUPS OF GENUS $g \geq 2$ AND REGULAR MAPS

We have already seen that for each $g \geq 2$, $H(\sqrt{2})$ has only finitely many normal subgroups with genus g. Therefore corresponding regular maps will also be finitely many. Those with genus $2 \leq g \leq 7$ are given in [2], [3] and [4].

Note that since $q = 4$, the only non–degenerate regular maps, we can have, are those of type $\{2, n\}$ or $\{4, n\}$. The former ones will correspond to $W_n(\sqrt{2})$ and having $g = 0$, will be regular n–gons on the sphere. Here we shall be interested in the latter type. Hence all regular maps will have type $\{4, 4\}$. We will denote the corresponding normal subgroup by [4, n]. Here n is the level of the subgroup.

6. PRINCIPAL CONGRUENCE SUBGROUPS OF $H(\sqrt{2})$

An important class of normal subgroups in $H(\sqrt{2})$ are the principal congruence subgroups. For modular group Γ, these are the groups

$$\Gamma(n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \mp \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod n \right\}.$$

Let now p be prime. The principal congruence subgroup $\Gamma_p(\sqrt{2})$ of level p is defined by

$$\left\{ M = \begin{bmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{bmatrix} \in H(\sqrt{2}) : M \equiv \mp I \bmod p \right\}.$$

Note that by the definition

$$\Gamma_p(\sqrt{2}) \lhd H_e(\sqrt{2}).$$

If 2 is a square mod p (i.e. $p \equiv \pm 1 \mod 8$), then $\sqrt{2} \in GF(p)$. Otherwise $\sqrt{2}$ lies in $GF(p^2)$ (quadratic extension of GF(p)). Then we have finite groups $H_p(\sqrt{2}) \leq PSL(2, p)$ or $PSL(2, p^1)$, and a homo morphism

$$\theta : H(\sqrt{2}) \longrightarrow H_p(\sqrt{2}).$$

Let $K_p(\sqrt{2}) := \operatorname{Ker} \theta$. Obviously

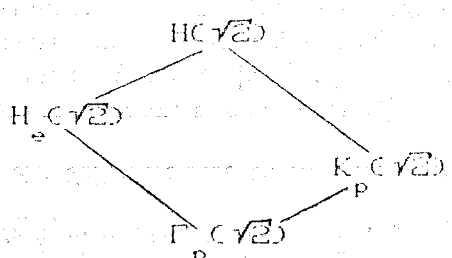
$$\Gamma_p(\sqrt{2}) \trianglelefteq K_p(\sqrt{2}).$$

It is not always the case that $K_p(\sqrt{2}) = \Gamma_p(\sqrt{2})$, e.g. if $p = 7$, then 2 is a square modulo 7. We know that $\Gamma_7(\sqrt{2}) \trianglelefteq He(\sqrt{2})$. Now the odd element

$$M = \begin{bmatrix} 5\sqrt{2} & 7 \\ 7 & 5\sqrt{2} \end{bmatrix} \in K_7(\sqrt{2})$$

as $\sqrt{2} = 3$ in GF(7), and therefore $K_7(\sqrt{2})$ contains an element which is not in $\Gamma_7(\sqrt{2})$. That is, the two congruence subgroups do not coincide for $p = 7$. Since $M \equiv I \mod 7$, $\Gamma_7(\sqrt{2})$ is a normal subgroup of $K_7(\sqrt{2})$ with index 2.

In general if 2 is a square mod p, we have



By [7], we find

$$H(\sqrt{2}) / K_p(\sqrt{2}) \cong PSL(2, p)$$

and therefore

$$H(\sqrt{2}) / \Gamma_p(\sqrt{2}) \cong C_2 \times PSL(2, p).$$

If 2 is not a square mod p, then $K_p(\sqrt{2}) = \Gamma_p(\sqrt{2})$ and we have

$$\begin{array}{c} H(\sqrt{2}) \\ | \\ He(\sqrt{2}) \\ | \\ K_p(\sqrt{2}) = \Gamma_p(\sqrt{2}). \end{array}$$

Therefore by [7],

$$H(\sqrt{2}) / \Gamma_p(\sqrt{2}) \cong PGL(2, p).$$

Theorem 5. $H(\sqrt{2}) / K_p(\sqrt{2}) \cong \begin{cases} PSL(2, p) & \text{if } p \equiv \pm 1 \mod 8, \\ PGL(2, p) & \text{if } p \equiv \pm 3 \mod 8, \\ C_2 & \text{if } p = 2, \end{cases}$

$$H(\sqrt{2}) / \Gamma_p(\sqrt{2}) \cong \begin{cases} C_2 \times PSL(2, p) & \text{if } p \equiv \pm 1 \mod 8, \\ PGL(2, p) & \text{if } p \equiv \pm 3 \mod 8, \\ D_4 & \text{if } p \equiv 2. \end{cases}$$

Therefore if p is an odd prime, then both congruence subgroups are free, while for $p = 2$, $K_2(\sqrt{2}) = H_e(\sqrt{2})$ and $\Gamma_2(\sqrt{2}) = W_4(\sqrt{2})$.

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