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ON THE RELATIONS BETWEEN lp NORMS FOR MATRICES

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ABSTRACT

The l_p norm, l_p operator norm and mixed l_{pq} norm of an mxn complex matrix $A = (a_{ij})$ are given by

$$|A|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p\right)^{1/p}$$

and

$$\|A\|_p = \max\left\{ |Ax|_p : x \in C^n, |x|_p = 1 \right\}$$

and

$$|A|_{pq} = \left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{m} |a_{ij}|^p \right)^{q/p} \right\}^{1/q}$$

respectively [2]. In this study we investigated the relations between l_p norms for matrices and we obtained some results.

STATEMENTS OF MAIN RESULTS

Lemma 1. [1] Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $x \in C^n$.

$$|x|_p \leq \lambda_{pq}(n) \; |x|_q \tag{1}$$

where

$$\lambda_{pq}(n) = \begin{cases} 1 & , p \geq q \\ n^{\frac{1}{p} - \frac{1}{q}} & , p \leq q \end{cases}$$

Theorem 1. Let A be n x n any complex matrix and let x be an n–vector satisfying $|x|_q = 1$. Then for $1 \leq p, q \leq \infty$

$$|A|_p \leq \xi_{pq}(n) \; \|A\|_q,$$

where

$$\xi_{pq}(n) = \begin{cases} n^{\frac{1}{p}} & , p \geq q \\ n^{\frac{2}{p} - \frac{1}{q}} & , p \leq q \end{cases}$$

Proof: Let us denote with $a_1, a_2, \ldots, a_n$ the columns of matrix A. Then by the definition of l_p matrix norm, we write

$$| A |_p = | (| a_1 |_p, | a_2 |_p, \ldots, | a_n |_p) |_p.$$

On the other hand, from Lemma 1 and since

$$a_j = Ae_j \quad j = 1, 2, \ldots, n$$

(where e_j is a suitable member of the standard basis of C^n.) we have

$$\begin{aligned} | A |_p &\leq | (\lambda_{pq}(n) \, | a_1 |_q, \lambda_{pq}(n) \, | a_2 |_q, \ldots, \lambda_{pq}(n) \, | a_n |_q) |_p \\ &= \lambda_{pq}(n) \, | (| a_1 |_q, | a_2 |_q, \ldots, | a_n |_q) |_p \\ &= \lambda_{pq}(n) \, | (| Ae_1 |_q, | Ae_2 |_q, \ldots, | Ae_n |_q) |_p \\ &\leq \lambda_{pq}(n) \max_{1 \leq j \leq n} | Ae_j |_q \, | (1, 1, \ldots, 1) |_p \\ &= \lambda_{pq}(n) \, n^{\frac{1}{p}} \max_{1 \leq j \leq n} | Ae_j |_q \\ &\leq \lambda_{pq}(n) \, n^{\frac{1}{p}} \max_{| x |_q = 1} | Ax |_q \\ &= \lambda_{pq}(n) \, n^{\frac{1}{p}} \, \| A \|_q. \end{aligned}$$

Whereas if we consider the definition of $\lambda_{pq}(n)$ in Lemma 1, then

for $p \geq q$, since $\lambda_{pq}(n) = 1$ in this case $\lambda_{pq}(n). \; n^{\frac{1}{p}}$ is equol to $n^{\frac{1}{p}}$.

Again for $p \leq q$ since $\lambda_{pq}(n) = n^{\frac{1}{p} - \frac{1}{q}}$, then $\lambda_{pq}(n). \; n^{\frac{1}{p}}$

is equal to $n^{\frac{2}{p} - \frac{1}{q}}$.

Thus the theorem is proved.

Theorem 2. Let A be any nxn complex matrix. Then for $1 \leq p \leq q \leq \infty$

$$\| A \|_p \leq n^{\frac{q-p}{pq}} \| A \|_q$$

Proof: Let z be an n–vector satisfying

$$| z |_p = 1, \quad | Az |_p = \| A \|_p.$$

Denoting $s = \frac{z}{| z |_q}$ we have $| s |_q = 1$. Hence using Lemma 1 we find

$$\begin{aligned}
\| A \|_p = | Az |_p &\leq n^{\frac{1}{p} - \frac{1}{q}} | Az |_q \\
&= n^{\frac{q-p}{pq}} | As |_q \, | z |_q \\
&\leq n^{\frac{q-p}{pq}} | As |_q \, | z |_p \\
&= n^{\frac{q-p}{pq}} | As |_q \\
&\leq n^{\frac{q-p}{pq}} \max_{| s |_q = 1} | As |_q \\
&= n^{\frac{q-p}{pq}} \| A \|_q.
\end{aligned}$$

Thus the proof is completed.

Theorem 3. Let A be nxn any complex matrix. Then for $1 \leq q \leq p \leq \infty$

$$\| A \|_p \leq \| A \|_q.$$

Proof: From Lemma 1, for $1 \leq q \leq p \leq \infty$ since

$$| Az |_p \leq | Az |_q$$

Therefore the proof is immediately seen.

Theorem 4. Let A be any nxn complex matrix. Then for $1 \leq p, q \leq \infty$

$$| A |_p \leq n^{\frac{1}{p}} | A |_{qp}.$$

Proof: Let us take $B = AE$ where E is nxn identity matrix. Thus from the product of matrices we have

$$B = (b_{ij}) = \left(\sum_{k=1}^{n} e_{ik}a_{kj}\right).$$

On the other hand using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we write

$$| b_{ij} | = | \sum_{k=1}^{n} e_{ik}a_{kj} | \leq \left\{ \sum_{k=1}^{n} | e_{ik} |^p \right\}^{\frac{1}{p}} \left\{ \sum_{k=1}^{n} a_{kj} |^q \right\}^{\frac{1}{q}}$$

Thus we obtain

$$| b_{ij} |^p = | \sum_{k=1}^{n} e_{ik}a_{kj} |^p \leq \left(\sum_{k=1}^{n} | e_{ik} |^p \right) \left(\sum_{k=1}^{n} |a_{kj} |^q \right)^{\frac{p}{q}}$$

Consequently, we find

$$| B |_p^p = \sum_{i,j=1}^{n} | b_{ij} |^p$$

$$\leq \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} | e_{ik} |^p \right) \left(\sum_{k=1}^{n} | a_{kj} |^q \right)^{\frac{p}{q}}$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} | e_{ik} |^p \right) \sum_{j=1}^{n} \left(\sum_{k=1}^{n} | a_{kj} |^q \right)^{\frac{p}{q}}$$

$$= n. | A |_{qp}^p$$

Whereas since $|B|_p = |A|_p$, we write

$$|A|_p \leq n^{\frac{1}{p}} |A|_{qp}.$$

So the theorem is proved.

Theorem 5. Let $A = (a_{ij})$ be nxn complex matrix. Then for $p \geq 1$

$$|A|_p^p \leq \sum_{i,j=1}^{n} \sum_{k=0}^{p} \binom{p}{k} |b_{ij}|^{p-k} |C_{ij}|^k$$

where $A = B + iC$ and $k = 0, 1, \ldots, p$.

Proof: We know that every complex matrix A can be written in the following form:

$$A = B + iC \text{ or for all } 1 \leq i, j \leq n, \; a_{ij} = b_{ij} + ic_{ij}.$$

On the other hand, since

$$|a_{ij}| \leq |b_{ij}| + |c_{ij}|,$$

for $p \geq 1$ we have

$$|a_{ij}|^p \leq (|b_{ij}| + |c_{ij}|)^p.$$

By Binominal expanded, we obtain

$$(|b_{ij}| + |c_{ij}|)^p = \binom{p}{0} |b_{ij}|^p + \binom{p}{1} |b_{ij}|^{p-1} |c_{ij}| + \ldots + \binom{p}{p} |c_{ij}|^p$$

$$= \sum_{k=0}^{p} \binom{p}{k} |b_{ij}|^{p=k} |c_{ij}|^k.$$

Therefore we get

$$|A|_p^p = \sum_{i,j=1}^{n} |a_{ij}|^p \leq \sum_{i,j=1}^{n} (|b_{ij}| + |c_{ij}|)^p$$

$$= \sum_{i,j=1}^{n} \sum_{k=0}^{p} \binom{p}{k} |b_{ij}|^{p-k} |c_{ij}|^k,$$

and so the theorem is proved.

REFERENCES

[1] GASTINEL, N.: Linear Numerical Analysis, Hermann, Paris, (1970).

[2] TASCI, D.: On a Conjecture by Goldberg and Newman, Linear Algebra and Its Applications (to appear).