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ON NEAR - LINEAR SPACES

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ABSTRACT

The goal of this work is to gathering together the general theory of the near–linear spaces and give some results on these spaces. As is known the many researchers are studying intensively on these topics.

1. DEFINITIONS AND BASIC PROPERTIES

A space S = (P, L) is a system of points P and lines L such that certain conditions or axioms are satisfied.

A near–linear space is a space S = (P, L) of points P and lines L such that

NL1 any line has at least two points, and

NL2 two points are on at most one line.

Near–linear spaces can be thought as the most general spaces. Also a near–linear space partial plane, a term introduced by M. Hall (194) [10].

By addition of some new axioms the other many near–linear spaces can be obtained from a given one.

In this work we examine polar spaces, linear spaces and as a special linear space, projective spaces and affine spaces.

For a line l_i, $v(l_i) = v_i$ will be the number of points on l_i. For a point p_j, $b(p_j) = b_j$ will be the number of lines on P_j.

Given, a system of points and lines, for any point p not on a line l, the connection number, $c(p, l)$, is the number of points on l which are connected to p by a line.

The material presented in the work is based on the notion of connection number in a near–linear space. In the near–linear spaces there are no restrictions on c(p, l). In the linear spaces, c(p, l) must always be the total number of points on l. In the polar spaces, c(p, l) must always be l or v(l).

The following definitions [1] are to show how can obtained a new near–linear space from a given one.

Let S = (P, L) be a near–linear space. We define R = (P′, L′) as follows. P′ is an arbitrary subset of P and L′ is the set of intersections $l \cap P'$ for any l in L with at least two points in P′. R is called a restriction of S and in particular, the restriction of S to P′.

Let S = (P, L) be a near–linear space. We define the dual (near–linear) space R = (P′, L′) of S as follows

$$P' = L$$

and any set of at least two lines of S which is the set of all lines through a fixed point of S is a line of L′, and these are the only lines. In brief $L' = \{ \{P_1, P_2, \ldots, P_m\} \mid P_i \in P',\ m \geq 2$ and $P_1, P_2, \ldots, P_m$ are all the lines of S incident with a fixed point.$\}$

A subspace of a near–linear space (P, L) is a set X of points of P such that whenever p and q are points of X which are on a line pq of L, then the entire line pq is in X.

All the restrictions, subspaces and dual space of a nearlinear space S are near–linear spaces. Also intersections of any numbers of subspaces is a near–linear space.

Let X be any set of points of a near–linear space S = (P, L). The closure $\langle X \rangle$ of the set $X \subset P$ is the intersection of all subspaces containing X.

We say that X generates its closure. Conversely, given a subspace R we say that X is a generating set for R if $\langle X \rangle = R$, so that, also, X generates R.

A basis of a near–linear space S is an independent subset of the points of S which generates S.

For any near–linear space S we define the dimension

$$\dim S := \inf \{ |B| - 1 \mid B \subset S \text{ and } \langle B \rangle = S \}.$$

A plane is a 2–dimensional subspace.

For all the bases of a near–linear space S it is not necessary to have same number of elements.

6– figures defined by Cater (1978) [8] for desarguesian projective planes, can be thought as an elementary nearl–linear space. They can used to justify above idea.

By a 6–figure in S, we mean any sequence of 6 distinct points (ABC, A′B′C′) such that ABC is a triangle, and A′∈BC, B′∈CA, C′∈AB. We say that (ABC, A′B′C′) is a menelaus 6–figure if A′,B′,C′ are colinear, and (ABC, A′B′C′) is a ceva 6–figures if AA′, BB′, CC′ are concurrent.

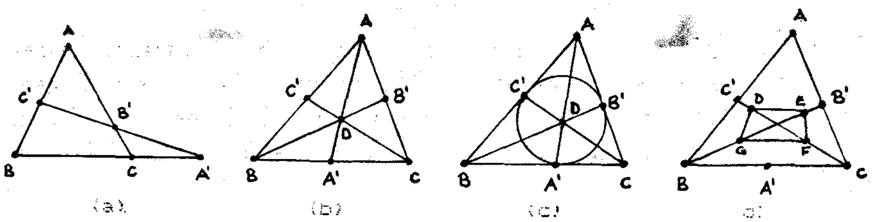

Figure 1.

Figure 1 a) is a menelaus 6–figure, b) is a ceva 6–figure, c) is, particularly called Fano plane. All the bases of these three near–linear spaces contain thtee elements. But the nearlinear space in d), have the bases {A, B, C} and {D, E, F, G}. Obviously the numbers of elements in these bases are different.

Fano plane has many properties and usually it is used as a sample space. Moreover Fano plane is only example that we can think as both menelaus 6–figure and ceva 6–figure.

Let $S = (P, L)$ and $S' = (P', L')$ be two near–linear spaces. A function $f\colon P \longrightarrow P'$ is called a linear function if $f(l) \in L'$ for all $l \in L$.

A linear function f is called an isomorphism if f^{-1} exist and it is a linear function. Then S and S′ are isomorphic spaces.

A collineation of $S = (P, L)$ is an isomorphism from S onto itself.

Let $S = (P, L)$ and $S' = (P', L')$ be two near–linear spaces. An embedding of near–linear space S into near–linear space S′ is a function f mapping P into P′ and L into L′ which is one–to–one on both points and lines, and such that $p \in l$ if and only if $f(p) \in f(l)$.

2. LINEAR SPACES AND POLAR SPACES

A linear space is a near–linear space in which any two points are on a line.

It is well known that Euclidean plane and extended real plane or the real projective plane are linear spaces.

Let $S = (P, L)$ be a Euclidean plane. For any line l let $[l]$ be the set of all lines parallel to l including l itself.

We construct a new space $S' = (P', L')$ as follows:

$$P' = P \cup \{[l] \mid l \in L\}$$
$$L' = \{\{l \cup [l] \mid l \in L\}, \{[l] \mid l \in L\}\}.$$

Where, the points of P' are those of P along with the parallel classes $[l]$. These classes will be called points at infinity. Each line of L gets a point at infinity added to it to make it a line of L'. L' gets in addition one new line, consisting of all the new points, which is called line at infinity.

Theorem 1. $S' = (P', L')$ as defined above is a linear space.

Proof: At first we must show that these new points and lines are well defined.

For this we must show that if $l' \in [l]$, then $[l'] = [l]$ and if $l' \in [l]$, then $[l'] \cap [l] = \varnothing$.

Let $l' \in [l]$. For every $x \in [l']$, $x = l'$ or $x \mid\mid l'$. Since $l' \in [l]$, $l' = l$ or $l' \mid\mid l$. From these we can write $x = l$ or $x \mid\mid l$. Therefore $x \in [l]$ and $[l'] \subset [l]$. Similarly we can see $[l] \subset [l']$. Thus $[l] = [l']$.

For $l' \notin [l]$ if were $[l'] \cap [l] \neq \varnothing$, then there was at least an $x \in [l'] \cap [l]$. Then $x \in [l'] \Rightarrow [x] = [l]$. From these follows $[l'] = [l]$. This is a contriction with $l' \notin [l]$.

Now we check up linear space's axioms.

For $l' \in L'$ let l' be an extention of a line of L. Then it contains infinetly many points.

If l' is the line at infinity, the numbers of points $[d]$ on l' are infinity also. Because in Euclidean plane each of lines, that incident with a fixed point, represent a parallel pencil $[l]$.

For $p', q' \in P'$ occours the following cases:

i) $p', q' \in P'$. In this case $p'q' \in L'$ is the only line connecting the points p' and q'.

ii) $p' \in P$ and $q' \in \{[l] \mid l \in L\}$. In this case the point P' is not on all the parallel lines beloŋgs to pencil $[l]$. Suppose that $p' \notin l$. Then in Euclidean plane there is only one parallel line to l from p'. The extentions of this line is $p'q'$.

iii) $p', q' \notin P$. In this case $p'q' = \{[l] \mid l \in L\}$.

There are many basic properties of numbers of points and lines of linear spaces that known. We do not tell about these but only one.

If S is a finite linear space with the exchange property. then any two bases have the same number of elements.

A hyperplane of a linear space S is a subspace which is covered by S. equivalently, it os a maximal proper subspece.

For example, in real 5–space, the hyperplanes are the 4–spaces; the hyperplanes of a line are points; the hyperplanes of a point are the empty set.

In general case it is not necessary that the dimension of a hyperplane be 1 less than the dimension of the space. But if S is a finite linear space with the exchange property and H is a hyperplane of S then $\dim H = \dim S - 1$.

A projective plane is a linear space in which

P1. any two lines meet,

P2. there exists a set of four points no three of which line on a single line.

The hyperplanes of a projective plane are precisely the lines.

Since dimension of a line is 1 and a projective plane has the exchange property it has dimension 2.

A projective hyperplane H is a proper subspace of a linear space S such that each line of S has a point in H.

It is easy to see that any projective hyperplane is a hyperplane.

Any projective plane Π has point and line regularity $k + 1$, say, $k \geq 2$ and $v = b = k^2 + k + 1$. We call k the order of the projective plane.

The linear space given in Theorem 1 is a projective plane of has infinite order. There are infinite many projective planes have finite order. Fano plane has 2 order and it is the smallest projective plane.

The one of the most important questions in geometry, does there exist a projective plane of order $n \geq 2$. But, the answer to this question in general is not known. It is known that there are unique projective planes of orders 2, 3, 4, 5, 7 and 8. There are at least four non isomorphic projective planes of order 9. The case of order 10 is still an unsolved problem. No one as yet exists.

From the Bruck–Rysers theorem we know that, if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and n is not the sum of the squares of two non negative intergers then there is no projective plane of order n. By this theorem there is no projective planes of orders 6, 14, 21, 22, 30, 33, 38, 42, 46,. . .

For every projective plane there is a algebraic representation which called ternary ring. There are very closed connection between projective plane Π and its ternary ring K. For example, Π is desarguesian if and only if K is a skew field and Π is pappian if and only if K is a field.

The algebraic representations provide many simplicities in the investigation of the projective planes. But the algebraic representation of the linear spaces do not yet obtain.

A subplane Π' of a projective plane Π is a projective plane whose point are a subset of the points of Π and in which each line is a subset of a line of Π.

We can determine relation between any projective plane and its subplanes by Bruck's theorem.

Theorem 2. (Bruck, 1963) [4]. If Π is a subplane of a finite projective plane Π and if the orders of Π and Π' are n and m respectively, then $\Pi \neq \Pi'$ implies $n = m^2$ or $m^2 + m \leq n$.

A projective space S is a linear space with the property that any two–dimensional subspace is a projective plane.

Theorem 3. Any subspace of a projective space is a projective space.

Proof: At first we will show that, if U is a subspace of V and V is a subspace of W, in a linear space S, then U is a subspace of W.

Because of definition of subspace, whever p and q are points of U which are on a line pq of V, then the entire line qp is in U. Since $U \subset V$. p and q are also points of V. Moreover since V is a subspace of W the entire line pq of W is in V. So U is a subspace of W.

Now we suppose that a subspace R of a projective space S has been given. Every plane of this subspace is also subplane of the projective space S and because of definition of projective space it is a projective plane. Therefore R itself is also a projective space.

Any two lines if a projective space have the same number of points and each projective space satisfies the exchange property.

An n–dimensional projective space of order k has $k^n + k^{n-1} + \ldots + k + 1$ points, $n \geq 0$.

An affine plane is a linear space $\mathcal{A}$ with the properties:

A1. any point p not line l is on precisely one line missing l, and.

A2. there exist a set of three non–collinear points.

The Euclidean plane is an affine plane.

Let Π be any projective plane and l be a line of Π. Then Π / l is an affine plane.

M. Hall [10] proved that any near–linear space in which there is a set of four points, no three collinear, is embeddable in a projective plane. Therefore each affine plane can be embedded in a projective plane.

There are many definition of an affine space. ae use Batten's definition.

An affine space is a projective space less a hyperplane.

Buekenhout [7] proved that a linear space with at least four points on every line is an affine space if all of its planes are affine.

Frank [9] proved following theorem and this theorem contains Buekenhout's theorem as a result for the case that every line contains at least five points.

Theorem 4. (Frank). Let (P, L) be a linear space with at least two lines. Assume that for every plane $\Sigma \subset P$, $(\Sigma, L(\Sigma))$ is the restriction of a projective plane $\Pi(\Sigma) = (\Sigma', L(\Sigma)')$ such that $3. | l \mid \Sigma | + 2 < |l|$ for all $l \in L(\Sigma)'$ with $l \cap \Sigma \neq \varnothing$. Then (P, L) is projectively embeddable.

Witte proved, in a manuscript remaning unpublished due to his death, that $b \leq n^2 + n + 1$ if and only if S is a near–pencil or embeds in a projective plane of order n. This was reproved by Stinson in 19882 [11].

In 1983, Stinson 'extended' this result to cover the case $b = n^2 + n + 1$. He showed that if $n^2 + 1 \leq v \leq n^2 + n + 1$ then $n \leq 3$ [12]. In this case there is only a handful of easily computable examples.

Since $v \leq b$ in any finite linear space the only cases not covered by Stinson [12] are $v = n^2$ and $v = n^2$ and $v = n^2 + n + 2$. However, if $v = b = n^2 + n + 2$, then it follows from the result in [6] that S is a near–pencil. The case $v = n^2$ was completed by Batten [2]. The case $n \leq v < (n + 1)^2$ and $b = n^2 + n + 3$ cas completed by Batten again [3]. He proved following result.

Theorem 5. (Batten). Let $S = (P, L)$ be a finite linear space an v points. Let $n \geq 10$ be the unique integer such that $n^2 \leq v < (n + 1)^2$. If $b = n^2 + n + 3$, and S is not a near–pencil, then S must be the following: an affine plane of order n less 0, 1, 2 or 3 points, with 3 additianal 'points at infinity', dnd 3 lines, each of size 2, an these 3 points.

A polar space $S = (P, L)$ is a near–linear space such that for every point p not in the line l, c $(p, l) = 1$ or v (l).

Any linear space is a polar space.

If some point of S is collinear with all points of S, we shall say that S is degenerate.

In particular, all linear spaces are degenerate polar spaces.

Non–degenerate polar spaces have particularly nice properties that are often lost in the degenerate case.

One can produce new polar space from given one.

The following lemma is only one of them.

Lemma. Any subspace of a polar space is again a polar space.

Proof: If X is a subspace of a polar space $S = (P, L)$, then for every p, $q \in X$, because of definition of subspace, $pq \subset X$. So X is a near–linear space.

Given a p point and a l line of X, Which $p \notin l$, each point r of l and point p can though also points of S. So the line through p and r (if exists) are same for X and S. Then c(P, L) is same in X and S. Since S is polar space $c(p, l) = 1$ or $v(l)$.

Here we only give the definitions and some properties of some near–linear spaces. In this occassion we should say that one can find many works on these problems in the recent literature.

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