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ON THE SECTIONAL CURVATURES OF TOTALLY REAL SUBMANIFOLDS IN S^6

M. ERDOĞAN — A. KILIÇ — A. ÖZEL

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SUMMARY

In this paper, we investigated the sectional curvatures of the submanifolds which are totally real in S^6.

INTRODUCTION

A 6 dimensional sphere S^6 does not admit any Kaehler structure. However a natural almost complex structure J can be defined on S^6. This structure on S^6 is nearly Kaehler, that is, it satisfies $(\nabla_X J)(X) = 0$, where ∇ is the Riemannian connection on S^6 and J is the almost complex structure of S^6 [5].

There are two types of submanifolds on S^6, those which are almost complex and those which are totally real. A Riemann manifold M isometrically immersed in S^6, is called a totally real submanifold of S^6 if $J(TM) \subset T^{\perp}M$ where $T^{\perp}M$ is the normal bundle of M in S^6, then we have $n = \dim M \leq 3$. In this paper we investigated the sectional curvatures of the submanifolds which are totally real in S^6.

1. PRELIMINARIES

Let $UM = \{X \in TM; \|X\| = 1\}$ be the unit tangent bundle of M. If M is two dimensional, consider the function $f: UM \to R$ defined by $f(v) = \langle h(V,V), JV \rangle$ which is clearly smooth, where h is the 2 nd fundamental form tensor of M. Suppose that f is not constant. The unit tangent bundle UM being compact, f attains its maximum at a tangent vector, say e_1. Then it is well known that $\langle h(e_1, e_1), Jy \rangle = 0$, for $y \in UM$ and $y \perp e_1$ [3].

Put $h(e_1, e_1) = a\, Je_1$, where a is a smooth function on M. Choose e_2 such that $\{e_1, e_2\}$ is a local orthonormal frame of M. Then we have the following expressions [1].

(*) $h(e_1, e_1) = a J e_1$, $h(e_2, e_2) = b j e_1 + c J e_2$, $h(e_1, e_2) = bJe_2$ where b, c are smooth functions on M.

Now assume that M is three dimensional. Let $x \in M$ and let us construct an orthonormal basis of T_xM in the following way [2]. Consider the function $f_1 : UM \to R$ defined by $f_1(v) = < h(V, V), J V >$ If f_1 attains an absolute maximum in u then $< h(u, u), J w > = 0$, for w orthogonal to u. Choose e_1 to be an absolute maximum of f_1. Then we consider the restriction of f_1 to $\{v \in UM_p \mid < v, e_1 > = 0\}$. We will denote this restriction of f_1 by f_2. If f_2 is identically zero, we choose e_2 as an eigenvector of A_{Je_1}, where A_{Je_2} is the shape operator with respect to Je_1. If f_2 is not identically zero, we take e_2 as an absolute maximum of f_2. Finally, we choose e_3 such that $G(e_1, e_2) = Je_3$. Then, the second fundamental form can be written as

$$\begin{aligned} h(e_1, e_1) &= a J e_1 \\ h(e_2, e_1) &= b J e_1 + c J e_2 \\ h(e_3, e_3) &= -(a + b) J e_1 - c J e_2 \\ h(e_1, e_2) &= b J e_2 + d J e_3 \\ h(e_1, e_3) &= -(a + b) J e_3 + dJ e_2 \\ h(e_2, e_3) &= dJ e_2 - c J e_3, \end{aligned}$$

where $a \geq d \geq 0$ and $b, c \in R$.

At this point we may express the following lemma which was proved in [2].

Lemma. If M is a 3-dimensional compact totally real submanifold of S^6, then for each point p of M, there exists an orthonormal basis $\{e_1, e_2, e_3\{$ of T_pM such that either

(i) $h(e_1, e_1) = h(e_2, e_2) = h(e_3, e_3) = 0$,
$h(e_1, e_2) = h(e_1, e_2) = h(e_2, e_3) = 0$,

or

(ii) $h(e_1, e_1) = (\sqrt{5}/2) J e_1$, $h(e_1, e_2) = (-\sqrt{5}/4) J e_2$,
$h(e_2, e_2) = (-\sqrt{5}/4) J e_1 + (\sqrt{10}/4) J e_2$, $h(e_1, e_3) = (-\sqrt{5}/4) Je_3$
$h(e_3, e_3) = (\sqrt{5}/4) J e_1 - (\sqrt{10}/4) J e_2$, $h(e_2, e_3) = (-\sqrt{10}/4) J e_3$,

or

(iii) $h(e_1, e_1) = (\sqrt{5}/2) J e_1$, $h(e_1, e_2) = (-\sqrt{5}/4) J e_3$,
$h(e_2, e_2) = (-\sqrt{5}/4) J e_1$, $h(e_1, e_2) = (-\sqrt{5}/4) J e_2$,
$h(e_3, e_3) = (-\sqrt{5}/4) J e_1$, $h(e_2, e_3) = 0$.

The lenght of the second fundamental form of M at point x is defined by

$$\| h_x \|^2 = \sum_{1 \leq i,j \leq n} \| h_x (e_i, e_j \|^2. \tag{1.1}$$

If P is a plane section of M at x, i.e. a two dimensional subspace of T_xM, then denote by K(P) the sectional curvature of M at P and by $h|_P$ the symmetric bilinear form from PxP to $T^{\perp}M$ obtained by restricting h_x to PxP. Let e_1, e_2 be any orthonormal basis of P. Then the Gauss curvature equation can be written as

$$K(P) = 1 + < h(e_1, e_1), h(e_2, e_2) > - \| h(e_1, e_2) \|^2. \tag{1.2}$$

and the length of $h|_p$ is $\| h|_p \|^2 = \sum_{1 \leq i,j \leq 2} \| h(e_i, e_j) \|^2$.

$$\| h|_p \|^2 = \| h(e_1, e_1) \|^2 + 2 \| h(e_1, e_2) \|^2 + \| h(e_2, e_2) \|^2 \tag{1.3}$$

2. RELATIONS BETWEEN SECTIONAL CURVATURES

Now, we may prove the following theorems providing some relations about the sectional curvatures of totally real submanifold M in S^6.

Theorem 1. Let M be an 2 or 3 dimensional totally real submanifold of S^6. If P is a plane section of M, then $K(P) \leq 1 + (1/2) \| h|_p \|^2 \leq 1 + (1/2) \| h \|^2$.

Proof: If M is 2 dimensional, then the sectional curvature K(P) coincides with the Gaussian curvature of M at P. For 2 dimensional case, it was proved by S. Deshmukh in [1] that the Gaussian curvature of M is 1, that is, M is totally geodesic. In this case, since $h|_p$ also coincides with h_x, we easily have $1 \leq 1 + (1/2) \|h|_p\|^2 = 1 + (1/2) \| h \|^2$. Now, let us give the proof of theorem for the case of dimension 3.

Let e_1, e_2 be an orthonormal basis of P. We will consider three cases in the Lemma. Case (i). From (1.2.) and (1.3.) we get

$$\| h \|^2 = 0, \quad \| h|_p \|^2 = 0 \text{ and so } K(P) = 1.$$

Case (ii). From (1.2.) and (1.3.) we get

$$K(P) = 1 + < (\sqrt{5}/2) \quad Je_1, \quad (-\sqrt{5}/4) \quad Je_2 + (\sqrt{10}/4) \quad Je_2 \geq$$
$$- \| <-\sqrt{5}/4) \, Je_2 \|^2 = 1/16$$

and

$$\| h \|^2 = \sum_{1 \leq i,j \leq 3} \| h_x (e_i, e_j) \|^2 = 95/16, \quad \| h|_p \|^2 = 45/16$$

and so

$1/16 \leq 1 + 45/32 \leq 1 + 95/32$, which proves the assertion.

Case (iii). From (1.2) and (1.3) we get

$K(P) = 1 + < (\sqrt{5}/2)\, Je_1,\ (\sqrt{5}/4)\, Je_1 > - \| (-\sqrt{5}/4)\, Je_2 \|^2 = 1/16$

and

$\| h \|^2 = \sqrt{50}/16,\quad \| h|_p \|^2 = 35/16$

and so

$1/16 \leq 1 + 35/32 \leq 1 + 50/32$, which proves the assertion.

Theorem 2. If M is a totally real minimal surface of S^6. Then, we have

$K(P) = 1-(1/2)\| h \|^2 \leq 1.$

Proof: If M is a minimal surface, then mean curvature vector of M is zero so from (*) we get $h(e_2, e_2) = -a\, J\, e_1$. Using this in (1.2) and (1.3) it follows that

$K(P) = 1 + < aJe_1, -aJe_1 > - < bJe_2, bJe_2 > = 1 - (a^2+b^2)$

and

$\| h \|^2 = 2\,(a^2 + b^2)$

and so

$K(P) = 1-(1/2)\, \| h \|^2 \leq 1.$

Remark to Theorem 2. In three dimensional case Theorem 2 is justified for the only case (i) and the other cases do not occur.

Theorem 3. If M is a totally real and also totally umbilic submanifold of S^6, then, wehave $K(P) = 1$.

Proof: If M is three dimensional, then only the case (i) occurs, so the proof for this case is clear. If M is a totally real and also totally umbilic surface, then by definition we write $h(e_2, e_2) = 0$ and $h(e_1, e_1) = h(e_2, e_2)$. From (*), it follows that $a = b = c = 0$, which imply $h(e_1, e_1) = h(e_2, e_2) = h(e_1, e_2) = 0$. Thus, from (1.2) we have $K(P) = 1$.

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