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CROSS-RATIOS OVER THE GEOMETRIC STRUCTURES WHICH ARE COORDINATIZED WITH ALTERNATIVE OR LOCAL ALTERNATIVE RINGS

BASRI ÇELIK — SULEYMAN ÇIFTÇI

Department of Mathematics, Uludağ University, Bursa / Turkey.

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ABSTRACT

In this note we summarize [3], [4] and [9] which are interested with cross ratio over alternative or local alternative rings. In [9], there is an algebraic expression which is not true [6], [7]. For this, we give some proofs by using formal calculations. And we give some basic results which are useful for calculations.

1. INTRODUCTION

When A is an alternative division ring with char $\neq 2$, the definition of cross–ratio over the projective line which is coordinatized by A is given by Schleiermacher (1965, [18]). In 1980, Ferrar [9] used this definition and did a similar work. Blunck (1991, [3]) "extended" this definition to the case of A having an arbitrary characteristic. In [6] and [7] definition of the cross–ratio is extended to the whole Mufang plane which is coordinatized by A.

More information about the geometric and algebraic preliminaries may be found in [10], [13], [15], [19] and [12], [16] respectively.

2. BASIC CONCEPTS

Let $(\mathcal{P}, \mathcal{L}, \in)$ be an incidence structure whose points and lines are denoted by P, Q, R, ... and l, m, n, ... respectively.

Definition. An incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ is called a projective plane if the following axioms are satisfied:

P1) Any two distinct points are incident with a unique line.

P2) Any two distinct lines are incident with at least one point.

P3) There exist a set of four poins, no three collinear.

Definition. Let $(\mathcal{P}, \mathcal{L}, \in)$ be incidence structure and "o" be equivalence relation on $\mathcal{P}$ and on $\mathcal{L}$. We called "o" as neighbouring. Then $M = (\mathcal{P}, \mathcal{L}, \in, O)$ is called projective Klingenberg plane (PK–plane), if the following axioms are satisfied:

PK1) Any two non–neighbouring points are incident with a unique line.

PK2) Any two non–neighbouring lines are incident with a unique point.

PK3) There is a projective plane $M^* = (\mathcal{P}^*, \mathcal{L}^*, \in)$ and an incidence structure epimorphism $\Psi: M \longrightarrow M^*$ such that the conditions $\Psi(P) = \Psi(Q) \Leftrightarrow P \square Q$, $\Psi(\ell) = \Psi(m) \Leftrightarrow \ell \square m$ hold for all $P, Q \in \mathcal{P}$, $\ell, m \in \mathcal{L}$.

In PK3, M^* is the canonic image of M.

If there exist a $Q \in \mathcal{P}$ such that $P \square Q$ and $Q \in \ell$ then we say that a point P is near a line ℓ and we denote this by $P \square \ell$.

In PK–planes, neighbouring points may have no, unique, or several joining lines. And neighbouring lines may have no, unique, or several intersection points.

Definition. Let M be a PK–plane.

(i) If any two points of M have at least one joining line then M is called punctally cohesive.

(ii) If any two lines of M have at least one intersection point then M is called lineary cohesive.

If M PK–plane is punctally and lineary cohesive than M is called cohesive.

A projective Hjelmslev plane is a cohesive PK–plane, such that any two points with exactly one joining line are nonneighbouring.

Now, we recall some notions for M PK–planes (cf. [2]):

If σ is an incidence structure automorphism such that preserving neighbour relation then σ is called a collineatinon of M. A centre and an axis of a collineation are defined as usual (cf. [17]). A (c, α)–collineation Φ is a collineation with center C and axis α. If $C \in \alpha$ then (C, α) –collineation Φ is called (C, α)–elation (or elation). M is called (C, α)–

transitive, if for all P, Q $\in \mathcal{P}$, P, Q $\blacksquare$ C, P, Q, $\blacksquare$ α, P, Q, C collinear, there is a (C, α)–collineation mapping P to Q.

Definition: Let M be a PK–plane. If M is (C, α)–transitive for all (C, α), C$\in\alpha$, then M is called a Moufang–Klingenberg plane (MK–plane).

If M is MK–plane the canonic image M* is a Moufang plane.

MK–planes are coordinatized with local alternative rings. In [2], MK–planes are coordinatized, using Duga's method (c.f. [8]). Before we are going to summarize their results we recall the definition of local alternative rings:

Definition. An alternative ring R with identity element 1 is called local, if the I of its non–units is an ideal.

It is shown ([2]) that for every local alternative ring there is a corresponding coordinate plane, which is an MK–plane, and conversely, that every MK–plane may be coordinatized by local alterative ring. Now, we are going to summarize their coordinatization (For more information about coordinatization of MK–planes se [2], [5]. [14]):

Let M be a MK–plane and (O, E, U, V) a basis of M i.e. its canonic image (Ψ(O), Ψ(E), Ψ(U), Ψ(V)) is a non–degenerate quadrangle in M*. Let ℓ = OE, W: = $\ell \cap$ UV, R: = {P$\in\mathcal{P}$| P$\in\ell$, P $\blacksquare$ W}, I: = {P$\in$R | P$\square$O} espesially o: = O, 1: = E, Let g_∞: = UV and if P$\in\ell$, P $\square$ g_∞ then, when x$\in$R, (x, x, 1) is taken as the coordinates of P. The points P$\in\mathcal{P}$ of M get their coordinates as follows:

(i) P $\blacksquare$ g_∞ $\Rightarrow$ P = (x, y, 1) where (x, x, 1) = PV $\cap \ell$, (y, y, 1) = PU $\cap \ell$.

(ii) P $\square$ g_∞, P $\blacksquare$ V $\Rightarrow$ P = (1, y, z) where (1, z, 1) = (PV $\cap$ UE) O $\cap$ EV, (1, y, 1) = OP $\cap$ EV.

(iii) P $\square$ V $\Rightarrow$ P = (w, 1, z) where (1, 1, z) = PU $\cap$ 1, (w, 1, 1) = OP $\cap$ EU where w, z$\in$I.

And lines g$\in\mathcal{L}$ of M get their coordinates as follows:

(i) g$\blacksquare$V $\Rightarrow$ g = [m, 1, p] where (1, m, 1) = (g $\cap g_\infty$) O $\cap$ EV, (o, p, 1) = g $\cap$ OV.

(ii) g$\square$V, g $\blacksquare$$g_\infty$ $\Rightarrow$ g = [1, n, p] where (n, 1, 1) = (g $\cap g_\infty$) O $\cap$ EU, (p, 0, 1) = g $\cap$ OU.

(iii) g $\square$ g_∞ $\Rightarrow$ g = [q, n, 1] where (1, o, q) = g $\cap$ OU, (o, 1, n) = g $\cap$ OV

where q, n$\in$I.

Now we are going to give the algebraic correspondings of the incidence relation:

$$(x, y, 1) \in [m, 1, p] \Leftrightarrow y = xm + p$$

$$(1, y, z) \in [m, 1, p] \Leftrightarrow y = m + zp$$

$$(w, 1, z) \notin [m, 1, p]$$

$$(x, y, z) \in [1, n, p] \Leftrightarrow x = yn + p$$

$$(w, 1, z) \in [1, n, p] \Leftrightarrow w = n + zp$$

$$(1, y, z) \notin [1, n, p]$$

$$(1, y, z) \in [q, n, 1] \Leftrightarrow z = q + yn$$

$$(w, 1, z) \in [q, n, 1] \Leftrightarrow z = wq + n$$

$$(x, y, 1) \notin [q, n, 1].$$

Therefore, the lines may be given as follows with the incidence points:

$$[m, 1, p] = \{(x, xm + p, 1) \mid x \in R\} \cup \{(1, zp, + m, z) \mid z \in I\}$$

$$[1, n, p] = \{(yn + p, y, 1) \mid y \in R\} \cup \{(zp + n, 1, z) \mid z \in I\}$$

$$[q, n, 1] = \{(1, y, yn + q) \mid y \in R\} \cup \{(w, 1, wq + n) \mid w \in I\}.$$

Now we can state;

Theorem 1. (cf. [2]) Let M be a MK–plane coordinatized as above. Then (R, + .) is a local alternative ring with I the ideal of nonunits. Neighbourhood in M is characterized by

$$(x_1, x_2, x_3) \square (y_1, y_2, y_3) \Leftrightarrow x_i - y_i \in I,\ i = 1, 2, 3$$

$$[x_1, x_2, x_3] \square [y_1, y_2, y_3] \Leftrightarrow x_i - y_i \in I,\ i = 1, 2, 3$$

Conversely, given a local alternative ring R one can construct an MK–plane M (R) over R.

3. CONJUGACY AND CROSS–RATIO IN LOCAL ALTERNATIVE RINGS

Let A be an alternative ring with identity element. On the $R := R(\varepsilon) = A + A\varepsilon$, $(\varepsilon^2 = 0)$ the operations (+) and (.) may be defined as follows:

$$(a_1 + a_2\varepsilon) + (b_1 + b_2\varepsilon) = a_1 + b_1 + (a_2 + b_2)\varepsilon$$

$$(a_1 + a_2\varepsilon)\,(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon$$

Then, there is a local alternative ring $I = A\varepsilon$ is ideal of non units (cf. [4]).

In this section, generalizing the well known ones e.g. [9] or [18], the definition of cross–ratio is given on the projective line which is coordinatized with the local alternative rings like as R.

In [9], the concept of cross–ratio over the projective line which is coordinatized by alternative division ring A is given as conjugacy class, i.e.

$$a \equiv b \Leftrightarrow \exists \text{ unit } c \text{ such that } a = c^{-1}bc.$$

Before the state the characterization of "$\equiv$" on R we want to give some concepts which are given in [9] for A:

A is a Cayley division algebra over its center Z which is equipped with an involutory anti–automorphism $K: x \longrightarrow \bar{x}$. The norm form $N: x \longrightarrow N(x) := x\bar{x}\ (= \bar{x}x)$ is a quadratic form which is multiplicative in the sense that $N(ab) = N(a)\,N(b)$ for all $a, b \in A$. The trace linear form on A is defined by $T: x \longrightarrow T(x) := x + \bar{x}$. The trace form is symetric ($T(ab) = T(ba)$) and associative ($T(a(bc)) = T((ab)c)$). Then the following characterization of conjugacy in propper alternative fields may be given as follows:

Lemma 1. Let A be non–associative alternative division ring. Then all $a, b \in A$ $a \equiv b \Leftrightarrow N(a) = N(b)$ and $T(a) = T(b)$.

More information about the norm and trace form are givenin [11] and [16].

By Lemma 1 "$\equiv$" is an equivalence relation over every altenative division ring A.

The concepts of involutory anti–automorphism, norm and trace form are may be extented over R as follows:

Let $x = x_1 + x_2\varepsilon \in R$. The map

$$K: R \longrightarrow R$$

$$x \longrightarrow K(x) := \bar{x} := \bar{x}_1 + \bar{x}_2\varepsilon$$

is involutory $Z(\varepsilon)$ $(= Z + Z\varepsilon)$–linear anti–automorphism over R. The norm form $N(x) := x\bar{x}\ (= \bar{x}x)$ is multiplicative and trace form $T(x) := x + \bar{x}$ is $Z(\varepsilon)$–linear.

Lemma 2. Let a, b, $c \in R$, $a = a_1 + a_2 \varepsilon$.

i) There is a symetric bilinear form f such that

$N(a) = N(a_1) + f(a_1, a_2)\varepsilon$.

ii) $T(a) = T(a_1) + T(a_2)\varepsilon$.

iii) The trace form is symetric and associative, i.e.

$T(ab) = T(ba)$, $T(a(bc)) = T((ab)c)$.

iv) $T(a) = T(\bar{a})$, $N(a) = N(\bar{a})$.

v) $f(a_1, a_2) = T(a_1 \bar{a}_2)$

vi) $N(a) = 0 \Leftrightarrow a \in I$. If $a \in R / I$ $\quad a^{-1} = N(a)^{-1} \bar{a}$.

iv) Since addition is commutative and K is an involutory

$$T(a) = a + \bar{a} = \bar{a} + a = \bar{a} + \bar{\bar{a}} = T(\bar{a})$$

$$N(a) = a\bar{a} = \bar{a}a = \bar{a}\bar{\bar{a}} = N(\bar{a})$$

v) $N(a) = a\bar{a} = (a_1 + a_2\varepsilon)(\bar{a}_1 + \bar{a}_2\varepsilon) = a_1\bar{a}_1 + (a_1\bar{a}_2 + a_2\bar{a}_1)\varepsilon$
$= N(a_1) + f(a_1, a_2)\varepsilon$

$$\Rightarrow f(a_1, a_2) = a_1\bar{a}_2 + a_2\bar{a}_1 = T(a_1\bar{a}_2) \ (= T(a_2\bar{a}_1))$$

(i), (ii), (iii) and (vi) are proved in [4].

Using together the properties $T(ab) = T(ba)$ and $T(a) = T(\bar{a})$ we obtain $f(a_1, a_2) = T(a_1\bar{a}_2)\, T = (a_2\bar{a}_1) = T(\bar{a}_1 a_2) = T(\bar{a}_2 a_1)$.

Now we can state the characterizatiom of "≡" over the R as follows:

Theorem 2. ([4]) Let R be non–associative, $a = a_1 + a_2\varepsilon$, $b = b_1 + b_2\varepsilon \in R$.

(i) If $a_1 \in Z$, then $a \equiv b \Leftrightarrow a_1 = b_1$ and $a_2 \equiv b_2$

(ii) If $a_1 \notin Z$, then $a \equiv b \Leftrightarrow N(a) = N(b)$ and $T(a) = T(b)$

Lemma 3. "≡" is an equivalence relation over R.

Proof: Let a, b, $c \in R$, $a = a_1 + a_2\varepsilon$, $b = b_1 + b_2\varepsilon$, $c = c_1 + c_2\varepsilon$.

Because of the definition "$a \equiv a$" and "$a \equiv b \Leftrightarrow b \equiv a$" is obvious. Let $a \equiv b$ and $b \equiv c$. Then there are two case:

i) If $a_1 \in Z$, then $a_1 = b_1 \in Z$, $b_1 = c_1$ i.e. $a_1 = c_1$ and $a_2 \equiv b_2$, $b_2 \equiv c_2$ i.e. $a_2 \equiv a_2$ (Lemma 1). Therefore $a \equiv c$.

ii) If $a_1 \notin Z$ then we have $N(a) = N(b)$, $T(a) = T(b)$, $N(b) = N(c)$, $T(b) = T(c)$. Therefore $N(a) = N(c)$, $T(a) = T(c)$. Hence $a \equiv c$.

In [4], it is shown that for all $a, b, c \in R$ $ab \equiv ba$ and $a(bc) \equiv (ab)c$.

An equivalence class of $x \in R$ is denoted by $[x]$. If $x \in A$, $[x]_A$ is denoted its conjugacy class in A. It can be shown that, by direct computation, $[x]_A = [x] \cap A$.

4. CROSS-RATIO OVER PROJECTIVE LINE

Let $\infty \notin A$. The projective line which is coordinatized by A is denoted by P(A). Therefore $P(A) = A \cup \{\infty\}$. Generalizing this over R, $P(R) = R \cup I^{-1}$ is obtained, where $I^{-1} := \{(a\varepsilon)^{-1} \mid a\varepsilon \in I\}$ consist of formal inverses of the non-units of R. On P(R), neighbour relation "o" is defined by;

$$x \square y :\Leftrightarrow (x, y \in I^{-1}) \text{ or } (x, y \in R,\ x-y \in I).$$

Using the fact that I is an ideal, it can be shown that neighbouring is an equivalance relation.

By putting $(0\varepsilon)^{-1} = 0^{-1} = \infty$, $P(A) \subseteq P(R)$ is obtained. The mapping $\Psi : P(R) \longrightarrow P(A)$, $\Psi(x_1 + x_2\varepsilon) = x_1$, $\Psi((a\varepsilon)^{-1}) = \infty$ is surjective and leaves "o" invariant. Also $x o y \Leftrightarrow \Psi(x) = \Psi(y)$. So $P(R) / \square \cong P(A)$ via the canonic epimorphism Ψ.

The operations with the elements of I^{-1} is as follows (cf. [4]):

Let $(a\varepsilon)^{-1} \in I^{-1}$, $c \in R$ and $q \in R / I$. Then,

$$(a\varepsilon)^{-1} + c := (a\varepsilon)^{-1} =: c + (a\varepsilon)^{-1}$$

$$q\,(a\varepsilon)^{-1} := (aq^{-1}\varepsilon)^{-1} \ (= a\,(q_1^{-1}\varepsilon)^{-1});\ q = q_1 + q_2\varepsilon$$

$$(a\varepsilon)^{-1}q := (q^{-1}a\varepsilon)^{-1} \ (= (q_1^{-1}a\varepsilon)^{-1})$$

$$((a\varepsilon)^{-1^{-1}}) = a\varepsilon$$

Other terms are not defined.

With the help of these rules, some special permutations (which are given in [9] over A) can be given over R:

(1): $t_\alpha: x \dashrightarrow x + \alpha$; $\alpha \in R$

(2): $l_q: x \dashrightarrow qx$; $q \in R / I$

(3): $r_q: x \dashrightarrow xq$; $q \in R / I$

(4): $i: x \dashrightarrow x^{-1}$

Let G be a group generated by all these permutations, i.e. $G = < t_\alpha, \ell_q, r_q, i > \ a \in R,\ q \in R / I$. Since $r_q = i \ell_{q-1}\, i$, G is generated by all t_α, ℓ_q, i.

Lemma 4. For the group $G = G(R)$, following statements are satisfied:

i) G preserves neighbourhood.

ii) G acts transitively on triples of pairwise nonneighbourint points of P(R).

Proof: i) is easy to show that t_α, l_α and i preserves neighbourhood.

ii) It is sufficient to see that there is a $\sigma \in G$ such that $\sigma(x) = 0$, $\sigma(y) = 1$ and $\sigma(z) = \infty$, for all pairwise non–neighbouring $x, y, z \in P(R)$. If σ is defined as follows then proof will be complated.

$$\sigma = \begin{cases} r_{(y-x)^{-1}}\, t_{-x} & ; \text{ if } z = \infty: = (0\varepsilon)^{-1} \\ r_{(y^{-1}-x^{-1})}\, t_{-x^{-1}}\, i & ; \text{ if } z = 0 \\ r_{((y-z)^{-1}-(x-z)^{-1})}\, t_{-(x-z)^{-1}}\, i t_{-z} & ; \text{ if } z \neq \infty,\ z \neq 0 \end{cases}$$

Definition. Let $a, b, c, d \in P(R)$ be pβirwise non–neighbourong. The cross–ratio (a, b; c, d) of the elements a, b, c, d is defined as a conjugacy class via:

$(a, b; c, d) = [((a-d)^{-1} (b-d)) ((b-c)^{-1} (a-c))]$; if $a, b, c, d \in R$

$(s^{-1}, b; c, d) = [((1 + ds) (b-d)) ((b-c)^{-1} (1-cs))]$; if $s^{-1} \in I^{-1}$, $b, c, d \in R$

$(a, s^{-1}; c, d) = [((a-d)^{-1} (1-ds)) ((1 + cs) (a-c))]$; if $s^{-1} \in I^{-1}$, $a, c, d \in R$

$(a, b; s^{-1}, d) = [((a-d)^{-1} (b-d) ((1 + sb) (1-sa))]$; if $s^{-1} \in I^{-1}$, $a, b, d \in R$

$(a, b; c, s^{-1}) = [((1 + sa) (1-sb)) ((b-c)^{-1} (a-c))]$; if $s^{-1} \in I^{-1}$, $a, b, c \in R$

The theorem which is given in [9] as Theorem 2 over alternative division ring A can be generalized over R i.e. every cross–ratio consist onlt of elements of $(R \setminus (\{0, 1\} + A\varepsilon)$. Conversely, the conjugacy class of any such element appears as a crossratio: Given three pairwise non–

neighbouring points a, b, c an element $r \in R/(\{0, 1\} + A\varepsilon)$, then there is a point $d \varnothing a, b, c$ with $[r] = (a, b; c, d)$ and if $r \in Z(\varepsilon)$ then d is unique (cf. [4]).

A permutation Φ of P(R) preserves cross ratios, if it preserves neigbourhood, and if

$$(a, b; c, d) = (\Phi(a), \Phi(b); \Phi(c), \Phi(d))$$

holds for all pairwise non–neigbouring $a, b, c, d \in P(R)$. The group of all such Φ is denoted by $S(R) = S$.

Now, we can state a theorem which is expressed cross–ratio in terms of addition, subtraction and inversion:

Theorem 3. Let $a, b, c, d \in R$ non–neighbouring pairvise points. Then

$$(a, b; c, d) = [((a-b)^{-1} - (a-d)^{-1})((a-b)^{-1} - (a-c)^{-1})^{-1}].$$

Proof: By Lemma 1, the multiplicative of N, and associativity of T $u' = ((a-d)^{-1}(b-d))((b-c)^{-1}(a-c))$ is conjugate to $u = (((a-d)^{-1}(b-d))(b-c)^{-1}(a-c)$. $\Rightarrow$

$u = (((a-d)^{-1}(b-d))\ (b-c)^{-1})\ (a-c) \ \Rightarrow\ (u(a-c)^{-1})\ (b-c) = (a-d)^{-1}(b-d)$. Thus,

$$((a-d)^{-1}(b-d))\ (a-b)^{-1} = ((u\ (a-c)^{-1})\ (b-c))\ (a-b)^{-1} \qquad (*)$$

First we compute the left hand side:

$$\begin{aligned}((a-d)^{-1}(b-d))\ (a-b)^{-1} &= ((a-d)^{-1}((a-d)-(a-b)))\ (a-b)^{-1}\\ &= ((a-d)^{-1}(a-d)-(a-d)^{-1}(a-b))\ (a-b)^{-1}\\ &= (1-(a-d)^{-1}(a-b))\ (a-b)^{-1}\\ &= (a-b)^{-1}-(a-d)^{-1}((a-b)\ (a-b)^{-1})\\ &= (a-b)^{-1}-(a-d)^{-1}.\end{aligned}$$

Putting this result in (*):

$$\begin{aligned}&(a-b)^{-1}-(a-d)^{-1} = ((u(a-c)^{-1})\ (b-c))\ (a-b)^{-1}\\ &\Rightarrow ((((a-b)^{-1}-(a-d)^{-1})\ ((a-b))\ (b-c)^{-1})\ (a-c) = u\\ &\Rightarrow [((((a-b)^{-1}-(a-d)^{-1})\ (a-b))\ (b-c)^{-1})\ (a-c)] = [u]\\ &\Rightarrow [((a-b)^{-1}-(a-d)^{-1}\ ((a-b))\ ((b-c)^{-1}(a-c)))] = [u]\end{aligned}$$

$\Rightarrow [((a-b)^{-1}-(a-d)^{-1}) (((a-c)^{-1}((a-c)-(a-b))) (a-b)^{-1})^{-1}] = [u]$

$\Rightarrow [((a-b)^{-1}-(a-d)^{-1}) ((a-b)^{-1}-((a-c)^{-1}(a-b)) (a-b)^{-1})^{-1}] = [u]$

$\Rightarrow [((a-b)^{-1}-(a-d)^{-1}) ((a-b)^{-1}-(a-c)^{-1})^{-1}] = [u] = [u'].$

In fact, this proof is given in [9] by using to the idea; "Moufang identities remain valid when one x is replaced by its inverce.". But it is not true (cf. [6], [7]). For this, we gave a formal proof above.

Lemma 5. For all a, b, c, d $\in$ P(R) pairwise non–neighbour ing elements (a, b; c, d) = (b, a; d, c).

Proof: There are threee cases:

(i) If a, b, c, d $\in$ R;

$(a, b; c, d) = [((a-d)^{-1}(b-d)) ((b-c)^{-1}(a-c))]$

$= [((b-c)^{-1}(a-c)) ((a-d)^{-1}(b-d))]$

$[b, a; d, c]$

(ii) If $s^{-1} \in I^{-1}$, b, c, d $\in$ R;

$(s^{-1}, b; c, d) = [((1 + ds) (b-d)) ((b-c)^{-1}(1-cs))]$

$= [((b-c)^{-1}(1-cs)) (((1 + ds) (b-d))]$

$= [b, s^{-1}; d, c]$

(iii) If $s^{-1} \in I^{-1}$, a, b, c$\in$R it is shown by similar way.

Theorem 4. G is a subgroup of S.

To prove this it is sufficient to see all t_α, r_q, i $\in$ S i.e. to see t_α r_q and i preserves cross–ratoio. If a, b, c, d$\in$R / I, t is proved in [9]. By Lemma 5 it must be seen only to the cases a$\in I^{-1}$ and d$\in I^{-1}$. These are may be seen by easy computation and some of these computations ara given in [4].

Lemma 6. Let a, b, c, d$\in$P(R) be pairwise non–neighbouring and $[x]^{-1}$ *is defined to be* $[x^{-1}]$ *and* $1-[x] = [1-x]$ *then*

$(a\ b; c, d)^{-1} = (b, a; c, d)$

$1-(a\ b; c\ d) = (a, c; b, d)$

$(a, b; c, d) = (b, a; d, c) = (c, d; a, b) = (d, c; b, a)$

Proof: By lemma 4–ii and Theorem 4 we can chose a $= \infty$, b $=$ o, c $=$ 1. Then this lemma is proved by easy computation.

Corollary. ([4]) Let (a, b, c, d) and (x, y, z, t) are pairwise non-neighbouring quadruples of P(R). Then

$(a, b; c, d) = (x, y; z, t) \Leftrightarrow \exists \gamma \in G \ni \gamma(a) = x, \gamma(b) = y, \gamma(c) = z, \gamma(d) = t.$

5. PERMUTATION GROUPS ON A LINE IN MK–PLANES.

Let M be a MK–plane which is coordinatized with respect to (O E, U, V), R the corresponding local alternative ring, I the ideal of its non–units. Consider a line

$$g = OV = [1, 0, 0] = \{(0, y, 1) \mid y \in R\} \cup \{(0, 1, z) \mid z \in I\}$$

Let $P(R) = R \cup I^{-1}$, $I^{-1} = \{z^{-1} \mid z \in I\}$. P(R) is identified with g as follows:

$$y \longleftrightarrow (0, y, 1), \qquad z^{-1} \longleftrightarrow (0, 1, z).$$

Definition. Let h, $k \in \mathcal{L}$, $C \in \mathcal{P}$, $C \,\square\,$ h, k. Then well–defined bijection

$$\sigma := \sigma(h, C, k): h \longrightarrow k \ni \sigma(X) = \overline{XC} \cap k$$

mapping h to k is called a perspectivity from h to k with centre C. A finite product of perspectivities is called a projectivity. The set of all projectivities mapping an any line g onto itself is a group which is denoted by $\mathcal{K}$ or $\mathcal{K}(g)$.

The following lemma is given in [5] like as [1]:

Lemma 7. The group $\mathcal{K}$ preserves the neighbour relation and transitively on the triples of pairwise non–geighbouring point of P(R). Morever, $\mathcal{K}$ is generated by the products of three prespectivities.

Definition 5. (cf. [5]) For every MK–planes $G \leq \mathcal{K}$.

6. ALGEBRAIC DESCRIPTION OF THE PROJECTIVITIES

In this section we are going to summarise some definitions and theorems which are given in [5].

Lemma 8. If M is lineary cohesive then every perspectivity is induced by an elation, and every projectivity is induced by a projective collineation.

Now we shall consider coordinatization of M with respect to arbitrary bases (O′, E′, U′, V′). Without lost of generality, we may always take the set R as the set of coordinates, and subset I as the coordinates of points neighbouring O′. Of course, R is again a local alternative ring, I the set of its non–units. If M is coordinatized with respect to the basis (O, E′, U, V) where $1 = OV \cap UE = OV \cap UE'$, then for the points $y \in g = P(R)$ with $y \in R$, we can always assume that the condition

$$(0, y, 1) = y = (0, y, 1)'$$

holds.

Lemma 9. Let $R' = (R, \oplus, o)$ be the coordinate alternative ring of M with respect to the basis (O, E′, U, V) where $g \cap UE' = 1 = g \cap UE$. Then there is a $q \in R / I$, such that $a \oplus b = a + b$, $aob = (aq)(q^{-1}b)$ hold for all $a, b \in R$.

By using the aboves the following theorem can be proved:

Theorem 6. Let M be a cohesive MK–plane such that the canonic image M* is different from the smallest projective plane. Then $\mathcal{K} = G$.

ÖZET

Bu çalışmada [3], [4] ve [9] da verilen çifte oran tanımları ışığında konunun geniş bir incelemesi yapılmış ve [9] daki bazı sonuçların hatalı bir cebirsel ifadeye dayalı olan ispatları yerine formal ispatları verilmiştir. Ayrıca hesaplamalarda çok kullanışlı olan bazı temel sonuçlar çıkarılmıştır.

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