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ON A CLASS OF MEROMORPHIC STARLIKE FUNCTIONS WITH POSITIVE COEFFICIENTS

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Let $T_m(A, B, z_0)$ denote the class of functions $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n$ $(a \geq 1,\ a_n \geq 0)$ regular and univalent in the disc $U' = \{z: 0 < |z| < 1\}$, satisfying

$$-z\frac{f'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \text{ for } z \in U', \text{ and}$$

$w \in E$ (where E is the class of analytic functions w with $w(0) = 0$ and $|w(z)| \leq 1$), where $-1 \leq A < B \leq 1, 0 \leq B \leq 1$ and $f'(z_0) = -\frac{1}{z_0^2}$ $(0 < z_0 < 1)$. In this paper, sharp coefficient estimates for the class $T_M(A, B, z_0)$ have been studied Radius of meromorphic convexity, integral transform of functions in $T_M(A, B, z_0)$ have been obtained. It is also proved that the class $T_M(A, B, z_0)$ is closed under convex linear combination. In the last part, the convolution problem of these functions have been studied.

1. INTRODUCTION

Let Σ denote the class of functions of the form

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U' = \{z: 0 < |z| < 1\}$ having a simple pole at the origin. Let Σ_S denote the class of functions in Σ which are univalent in U and $\Sigma^*(\rho)$ be the subclass of functions $f(z)$ in Σ satisfying the condition

$$\operatorname{Re}\left\{-z\frac{f'(z)}{f(z)}\right\} < \rho. \tag{1.2}$$

Functions in $\Sigma^*(\rho)$ are called meromorphically starlike functions of order ρ.

The class $\Sigma^*(\rho)$ have been extensively studied by Pommerenke [5], Clunie [1], Kaczmarski [3], Royster [6] and others.

Let Σ_M denote the subclass of functions in Σ_s of the form $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ with $a_n \geq 0$ and let $\Sigma^*_M(\rho) = \Sigma_M \cap \Sigma^*(\rho)$.

Juneja and Reddy [2] have obtained certain interesting results for functions in $\Sigma^*_M(\rho)$. Since much work has not been done for meromorphic univalent functions, we introduce following class of functions:

Let T_M denote the class of functions $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n$ $(a \geq 1,\ a_n \geq 0)$ (The $a \geq 1$ is necessary, see Nehari [4, Ex. 8, p. 238]) regular and univalent in the disc $U' = \{z: 0 < |z| < 1\}$. Let $T_M(A, B)$ denote the subclass of functions in T_M satisfying the condition

$$-\frac{z f'(z)}{f(z)} \propto \frac{1 + Az}{1 + Bz},\quad z \in U'$$

where $\propto$ denote subordination and A and B are fixed numbers $-1 \leq A < B \leq 1$, $0 \leq B \leq 1$. Then by definition of subordination

$$-z\frac{f'(z)}{f(z)} = \frac{1 + A\,w(z)}{1 + B\,w(z)},\ \text{for some } z \in U',\ w \in E \tag{1.3}$$

where E is the class of analytic functions w with $w(0) = 0$ and $|w(z)| \leq 1$. Also $T_M(A, B, z_0)$ denote the subclass of functions in $T_M(A, B)$ satisfying $f'(z_0) = -\frac{1}{z_0^2}$ (where $0 < z_0 < 1$).

In this chapter, we obtain sharp coefficient estimates for the class $T_M(A, B, z_0)$. Radius of meromorphic convexity, integral transform of functions in $T_M(A, B, z_0)$ have been studied. It is also shown that the class $T_M(A, B, z_0)$ is closed under convex linear combination. In the last part, the convolution problem of these functions have been studied.

2. MAIN RESULTS

In this section we prove our main results.

Theorem 2.1. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ be regular in and belongs to $T_M(A, B)$ if and only if

$$\sum_{n=1}^{\infty} \{n(1+B)+A+1\} a_n \leq B-A. \tag{2.1}$$

Proof: Consider the expression

$$H(f, f') = |z f'(z) + f(z)| - |B z f'(z) + A f(z)|. \tag{2.2}$$

Replacing f and f′ by their series expansions we have, four $0 < |z| = r < 1$,

$$H(f, f') = \left| \sum_{n=1}^{\infty} (n+1) a_n z^n \right| - \left| (A-B) \frac{1}{z} + \sum_{n=1}^{\infty} (A+Bn) a_n z^n \right|$$

$$\leq \sum_{n=1}^{\infty} (n+1) a_n r^n - (B-A) \frac{1}{r} + \sum_{n=1}^{\infty} (A+Bn) a_n r^n,$$

or

$$r H(f, f') \leq \sum_{n=1}^{\infty} \{n(1+B)+A+1\} a_n r^{n+1} - (B-A).$$

Since this holds for all r, $0 < r < 1$, making $r \to 1$, we have

$$H(f, f') \leq \sum_{n=1}^{\infty} \{n(1+B)+A+1\} a_n - (B-A) \leq 0, \tag{2.3}$$

in view of (2.1). From (2.2), we thus have

$$\left| \frac{z \frac{f'(z)}{f(z)} + 1}{B z \frac{f'(z)}{f(z)} + A} \right| \leq 1.$$

Hence $f \in T_M(A, B)$.

Conversely, let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and

$$\left| \frac{z \frac{f'(z)}{f(z)} + 1}{B z \frac{f'(z)}{f(z)} + A} \right| \leq 1$$

or

$$\left| \frac{\sum_{n=1}^{\infty} (n+1)\, a_n z^n}{(A-B)\,\frac{1}{z} + \sum_{n=1}^{\infty} (A+Bn)\, a_n z^n} \right| \leq 1$$

or

$$\left| \frac{\sum_{n=1}^{\infty} (n+1)\, a_n z^{n+1}}{(B-A) - \sum_{n=1}^{\infty} (A+Bn)\, a_n z^{n+1}} \right| \leq 1$$

Since Re $(z) \leq |z|$

$$\text{Re} \left\{ \frac{\sum_{n=1}^{\infty} (n+1)\, a_n z^{n+1}}{(B-A) - \sum_{n=1}^{\infty} (A+Bn)\, a_n z^{n+1}} \right\} \leq 1$$

choosing $z = r$ with $0 < r < 1$, we get

$$\frac{\sum_{n=1}^{\infty} (n+1)\, a_n r^{n+1}}{(B-A) - \sum_{n=1}^{\infty} (A+Bn)\, a_n r^{n+1}} \leq 1. \qquad (2.4)$$

Let $S(r) = (B-A) - \sum_{n=1}^{\infty} (A+Bn)\, a_n r^{n+1}$.

$S(r) \neq 0$ for $0 < r < 1$, $S(r) > 0$ for sufficiently small values of r and $S(r)$ is continuous for $0 < r < 1$. Hence $S(r)$ can not be negative for any value of r such that $0 < r < 1$. Upon clearing the denominator in (2.4) and letting $r \to 1$ we get

$$\sum_{n=1}^{\infty} (n+1)\, a_n \leq (B-A) - \sum_{n=1}^{\infty} (n+1)\, a_n$$

or

$$\sum_{n=1}^{\infty} \{n(1+B) + A + 1\}\, a_n \leq (B-A).$$

Hence the Theorem.

Theorem 2.2. Let $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n$. If f is regular in

U' and satisfies $f'(z) = -\frac{1}{z_0^2}$, then $f \in T_M(A, B, z_0)$ if and only if

$$\sum_{n=1}^{\infty} [\{n(1+B) + A + 1\} - n(B-A) z_0^{n+1}] a_n \leq (B-A), \; a_n \geq 0. \quad (2.5)$$

The result is sharp.

Proof: From Theorem 4.2.1, we know that a function $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ regular in U' satisfies

$$\left| \frac{z \frac{g'(z)}{g(z)} + 1}{B z \frac{g'(z)}{g(z)} + A} \right| < 1, \; z \in U,$$

if and only if

$$\sum_{n=1}^{\infty} \{n(1+B) + A + 1\} b_n \leq (B-A).$$

Applying that result to the function $g(z) = f(z)/a$, we find that f satisfies (1.1) if and only if

$$\sum_{n=1}^{\infty} \{n(1+B) + A + 1\} a_n \leq (B-A) a. \quad (2.6)$$

Since $f'(z_0) = -\frac{1}{z_0^2}$, we also have from the representation of $f(z)$ that

$$a = 1 + \sum_{n=1}^{\infty} n a_n z_0^{n+1}.$$

Putting this value of a in the inequality (2.6), we have the required result.

For attaining the equality in (2.5), we choose the function

$$f(z) = \frac{\{n(1+B) + 1 + A\} \frac{1}{z} + (B-A) z^n}{\{n(1+B) + 1 + A\} - n(B-A) z_0^{n+1}}. \quad (2.7)$$

From (2.7), we have

$$a_n = \frac{B-A}{\{n(1+B)+1+A\} - n(B-A)\, z_0^{n+1}},$$

or

$$[\, \{n(1+B)+1+A\} - n(B-A)\, z_0^{n+1}\,]\, a_n = (B-A),$$

and

$$a = 1 + \sum_{n=1}^{\infty} n\, a_n\, z_0^{n+1}$$

$$= 1 + \frac{n(B-A)\, z_0^{n+1}}{\{n(1+B)+1+A\} - n(B-A)\, z_0^{n+1}}$$

$$= \frac{\{n(1+B)+1+A\}}{\{n(1+B)+1+A - \{n(B-A)\, z_0^{n+1}} > 1.$$

Theorem 2.3. If $f \in T_M(A, B, z_0)$, then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < R$, where

$$R = \underset{n>1}{\text{Inf.}} \left[\frac{(1-\delta)\, \{n(1+B)+1+A\}}{n(n+2-\delta)(B-A)} \right]^{1/(n+1)}$$

This result is sharp for each n for functions of the form (2.7).

Proof: In order to establish the required result, it suffices to show that

$$\left| 2 + \frac{z f''(z)}{f'(z)} \right| \leq 1 - \delta$$

or

$$\left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| \leq 1 - \delta$$

and

$$\left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| = \frac{\sum_{n=1}^{\infty} \frac{n(n+1)}{a}\, a_n\, |z|^{n+1}}{1 - \sum_{n=1}^{\infty} \frac{n}{a}\, a_n\, |z|^{n+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} n(n+2-\delta)\, a_n\, |z|^{n+1} \leq a(1-\delta).$$

Since $a = 1 + \sum_{n=1}^{\infty} n\, a_n\, z_0^{n+1}$, the above inequality can be written as

$$\sum_{n=1}^{\infty} \frac{n\,[(n+2-\delta)\,|z|^{n+1} - (1-\delta)\,z_0^{n+1}]}{(1-\delta)}\, a_n \leq 1. \qquad (2.8)$$

Also by Theorem 2.2, we have

$$\sum_{n=1}^{\infty} \frac{\{n\,(1+B)+1+A\} - n\,(B-A)\,z_0^{n+1}}{(B-A)}\, a_n \leq 1.$$

Hence (2.8) will be satisfied if

$$\frac{n\,[(n+2-\delta)\,|z|^{n+1} - (1-\delta)\,z_0^{n+1}]}{(1-\delta)} \leq$$

$$\frac{\{n\,(1+B)+1+A\} - n\,(B-A)\,z_0^{n+1}}{(B-A)}, \text{ for each } n = 1, 2, \ldots$$

$$|z| < \left[\frac{(1-\delta\ \{n\,(1+B)+1+A\}}{n\,(n+2-\delta)\,(B-A)}\right]^{1/(n+1)},$$

for each $n = 1, 2, \ldots$

This completes the proof of theorem. Sharpness follows if we take the same extremal function for which Theorem 2.2 is sharp.

Theorem 2.4. If $f \in T_M(A, B, z_0)$, then the integral transform

$$F(z) = c \int_0^1 u^c\, f(uz)\, du, \text{ for } 0 < c < \infty,$$

is in $T_M(A', B', z_0)$, where

$$\frac{1+B'}{B'-A'} \leq \frac{(A+B+2)(c+2)+(B-A)\,c}{2c\,(B-A)} - \frac{z_0^2}{c}.$$

The result is sharp for the function

$$f(z) = \frac{(A+B+2)\,\frac{1}{z} + (B-A)\,z}{(A+B+2) - (B-A)\,z_0^2}.$$

Proof: Let $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n \in T_M(A, B, z_0)$,

then

$$F(z) = c \int_0^1 u^c \left[\frac{a}{uz} + \sum_{n=1}^{\infty} a_n (u^n z^n)\right] du$$

$$= c \int_0^1 \left[u^{c-1} \cdot \frac{a}{z} + \sum_{n=1}^{\infty} a_n (u^{n+c} z^n)\right] du$$

$$= c \left[\frac{u^c}{c} \cdot \frac{a}{z} + \sum_{n=1}^{\infty} a_n \frac{u^{n+c+1}}{(n+c+1)} z^n\right]_0^1$$

$$= c \left[\frac{a}{cz} + \sum_{n=1}^{\infty} \frac{a_n z^n}{(n+c+1)}\right]$$

$$= \frac{a}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{[\{n(1+B')+1+A'\} - n(B'-A') z_0^{n+1}]\, c}{(B'-A')(n+c+1)} a_n \leq 1. \quad (2.9)$$

Since $f \in T_M(A, B, z_0)$ implies that

$$\sum_{n=1}^{\infty} \frac{\{n(1+B)+1+A\} - n(B-A) z_0^{n+1}}{(B-A)} a_n \leq 1,$$

(2.9) will be satisfied if

$$\frac{[\{n(1+B')+1+A'\} - n(B'-A') z_0^{n+1}]\, c}{(B'-A')(n+c+1)}$$

$$\leq \frac{\{n(1+B)+1+A\} - n(B-A) z_0^{n+1}}{(B-A)}$$

for each n,

$$\frac{n(1+B')+1+A'}{(B'-A')} \leq \frac{\{n(1+B)+1+A\}(n+c+1)}{c(B-A)} - \frac{n(n+1)}{c} z_0^{n+1}$$

or

$$\frac{1+B'}{B'-A'} \leq \frac{\{n(1+B)+1+A\}(n+c+1)+(B-A)c}{(n+1)(B-A)c} - \frac{n}{c} z_0^{n+1}. \tag{2.10}$$

The right hand side of (2.10) is an increasing function of n, therefore putting $n = 1$ in (2.10) we get:

$$\frac{1+B'}{B'-A'} \leq \frac{(A+B+2)(c+2)+(B-A)c}{2c(B-A)} - \frac{z_0^2}{c}.$$

Hence the theorem.

Theorem 2.5. Let γ be a real number such that $\gamma > 1$. If $f \in T_M(A, B, z_0)$, then the function F defined by

$$F(z) = \frac{(\gamma-1)}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)\, dt$$

also belongs to $T_M(A, B, z_0)$.

Proof: Let $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n$. Then from the representation of $F(z)$, it follows that

$$F(z) = \frac{a}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

where

$$b_n = \frac{\gamma-1}{\gamma+n} a_n.$$

Therefore

$$\sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A) z_0^{n+1}] b_n$$

$$= \sum_{n=1}^{\infty} \left[\frac{\gamma-1}{\gamma+n}\right] [\{n(1+B)+1+A\} - n(B-A) z_0^{n+1}] a_n$$

$$\leq \sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A) z_0^{n+1}] a_n$$

$$\leq (B-A), \text{ by Theorem 2.2.}$$

Hence $F(z) \in T_M(A, B, z_0)$.

Theorem 2.6 Let $f(z) = \frac{1}{z}$ and

$$f_n(z) = \frac{\{n(1+B)+1+A\}\frac{1}{z} + (B-A)z^n}{\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}},$$

$$n = 1, 2, 3, \ldots$$

Then $h \in T_M(A, B, z_0)$ if and only if it can be expressed in the form

$$h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where

$$\lambda \geq 0 \text{ and } \lambda + \sum_{n=1}^{\infty} \lambda_n = 1.$$

Proof: Let us suppose that

$$h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

$$= \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

where

$$a = \lambda + \sum_{n=1}^{\infty} \frac{\{n(1+B)+1+A)\}\lambda_n}{\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}}$$

and

$$a_n = \frac{(B-A)\lambda_n}{\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}}.$$

Then, it is easy to see that $f'(z_0) = -\frac{1}{z_0^2}$ and the condition (2.5) is satisfied. Hence $h \in T_M(A, B, z_0)$.

Conversely let $h \in T_M(A, B, z_0)$, and

$$h(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n.$$

Then, from (2.5), it follows that

$$a_n \leq \frac{(B-A)}{\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}},$$

$$(n = 1, 2, 3, \ldots).$$

Setting

$$\lambda_n = \left[\frac{\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}}{(B-A)}\right] a_n$$

and

$$\lambda = 1 - \sum_{n=1}^{\infty} \lambda_n,$$

we have

$$h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

This completes the proof of theorem.

Theorem 2.7. Let $f_j(z) = \frac{a_j}{z} + \sum_{n=1}^{\infty} a_{nj} z^n$, $j = 1, 2, \ldots, m$.

If $f_j \in T_M(A, B, z_0)$ for each $j = 1, 2, \ldots, m$, then the function $h(z) = \frac{b}{z} + \sum_{n=1}^{\infty} b_n z^n$ also belongs to $T_M(A, B, z_0)$ where

$$b = \sum_{j=1}^{m} \lambda_j a_j, \; b_n = \sum_{j=1}^{m} \lambda_j a_{nj}, \; (n = 1, 2, \ldots, m),$$

$$\lambda_j \geq 0 \text{ and } \sum_{j=1}^{m} \lambda_j = 1.$$

Proof: Since $f_j \in T_M(A, B, z_0)$, then

$$\sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}] \, |a_{nj}| \leq (B-A)$$

$$j = 1, 2, \ldots, m.$$

Therefore

$$\sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}] \, b_n$$

$$= \sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}] \sum_{j=1}^{m} \lambda_j a_{nj}$$

$$= \sum_{j=1}^{m} \lambda_j \sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}] a_{nj}$$

$$\leq \sum_{j=1}^{m} \lambda_j (B-A) = (B-A).$$

Hence by Theorem 2.2, $h \in T_M(A, B, z_0)$.

Theorem 2.8. If $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n \in T_M(A, B, z_0)$ and

$g(z) = \frac{b}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $b_n \leq 1$ for $n = 1, 2, \ldots$, then

$f * g \in T_M(A, B, z_0)$.

Proof: Let $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{b}{z} + \sum_{n=1}^{\infty} b_n z^n$,

then for convolution of functions f and g we can write

$$\sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}] a_n b_n$$

$$\leq \sum_{n=1}^{\infty} [\{n(1+B)+1+A\} - n(B-A)z_0^{n+1}] a_n,$$

because $b_n \leq 1$.

$$\leq (B-A), \text{ by } (2.5).$$

Hence, by Theorem 2.2 $f * g \in T_M(A, B, z_0)$.

Note: It will be of interest to find some other convolution results analogous to those of Juneja and Reddy [2].

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