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ON THE FOCAL SURFACES OF THE CONGRUENCES GENERATED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH SOME SURFACES

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ABSTRACT

In this paper, the focal surfaces of the congruences derived in [1] and [3] have been investigated and correspondences between them have been explained.

1. INTRODUCTION

Let a surface $\vec{x}$ be referred to its lines of curvatures. The congruences generated by the instantaneous screwing axes $\vec{G}$, $\overset{\rightrightarrows}{G}$ of the moving trihedrons connected with these lines are respectively,

$$\left\{\begin{array}{ll} \vec{y} = \vec{r} + t\vec{g}, & \vec{r} = \vec{x} + \dfrac{1}{r}\vec{\xi} \\ \\ \overset{\rightrightarrows}{y} = \overset{\rightrightarrows}{r} + \bar{t}\overset{\rightrightarrows}{g}, & \overset{\rightrightarrows}{r} = \vec{x} + \dfrac{1}{\bar{r}}\vec{\xi} \end{array}\right. \qquad (1.1)$$

[1]. In case $\vec{y}$ and $\overset{\rightrightarrows}{y}$ are normal congruences, let the surfaces generating these, be $\vec{z}$ and $\overset{\rightrightarrows}{z}$. And let these surfaces be referred to their lines of curvature. The congruence generated by the instantaneous screwing axis $\overset{\rightrightarrows}{G}{}^*$ of the moving trihedron connected with the lines of curvature $u = \text{const.}$ of $\vec{z}$ are

$$\overset{\rightrightarrows}{y}{}^* = r + \bar{t}^* \overset{\rightrightarrows}{g}{}^*, \quad \vec{r} = \vec{z} + \frac{1}{\bar{b}}\vec{n}. \qquad (1.2)$$

And the congruence generated by the instanteneous screwing axis $\overset{\rightrightarrows}{G}^{**}$ of the moving trihedron connected with the lines of curvature $v = \text{const.}$ of $\overset{\rightrightarrows}{z}$ are

$$\overset{\rightrightarrows}{y}^{**} = \overset{\rightrightarrows}{r} + \bar{t}^{**}\,\overset{\rightrightarrows}{g}^{**}, \quad \overset{\rightrightarrows}{r} = \overset{\rightrightarrows}{z} + \frac{1}{\beta}\overset{\rightrightarrows}{n} \tag{1.3}$$

[3].

2. THE PROPERTIES OF THE FOCAL SURFACES OF THE CONGRUENCES $\vec{y}$, $\overset{\rightrightarrows}{y}$, $\overset{\rightrightarrows}{y}^{*}$, $\overset{\rightrightarrows}{y}^{**}$

Since $\vec{p}$, $\vec{k}$ are the focal surfaces of the congruence $\vec{y}$; $\overset{\rightrightarrows}{p}$, $\overset{\rightrightarrows}{k}$ of $\overset{\rightrightarrows}{y}$; $\overset{\rightrightarrows}{p}^{*}$, $\overset{\rightrightarrows}{k}^{*}$ of $\overset{\rightrightarrows}{y}^{*}$; and $\overset{\rightrightarrows}{p}^{**}$, $\overset{\rightrightarrows}{k}^{**}$ of $\overset{\rightrightarrows}{y}^{**}$ [3], to *investigate* considering the cases where they coincide and refer to their lines of curvature, first we may write the moving trihedrons (DARBOUX's trihedrons) connected with a common point before calculating their first and second fundamental forms.

1) Since the moving trihedron connected with the point $\vec{x}$ of the line of curvature $v = \text{const.}$ on the surface $\vec{x}\,(u, v)$ is $(\vec{x}_1, \vec{x}_2, \vec{\xi})$, the trihedrons connected with the focal points corresponding to ρ_{II} of the focal surfaces $\vec{p}$, $\overset{\rightrightarrows}{p}^{*}$, $\overset{\rightrightarrows}{k}^{**}$ belonging to the congruences $\vec{y}$, $\overset{\rightrightarrows}{y}^{*}$, $\overset{\rightrightarrows}{y}^{**}$ and coinciding with the center surface $\vec{r}$ of the surface $\vec{x}$, are respectively,

$$(\vec{\xi}, \vec{x}_2, -\vec{x}_1), \quad (\vec{\xi}, \vec{x}_2, -\vec{x}_1), \quad (-\vec{\xi}, -\vec{x}_2, -\vec{x}_1).$$

2) Since the moving trihedron connected with the point $\vec{x}$ of the line of curvature $u = \text{const.}$ on the surface $\vec{x}\,(u, v)$ is $(\vec{x}_2, -\vec{x}_1, \vec{\xi})$, the trihedrons connected with the focal points corresponding to $\bar{\rho}_{II}$ of the focal surfaces $\overset{\rightrightarrows}{p}$, $\overset{\rightrightarrows}{k}^{*}$, $\overset{\rightrightarrows}{p}^{**}$ belonging to the congruences $\overset{\rightrightarrows}{y}$, $\overset{\rightrightarrows}{y}^{*}$, $\overset{\rightrightarrows}{y}^{**}$ and coinciding with the center surface $\overset{\rightrightarrows}{r}$ of the surface $\vec{x}$, are respectively,

$$(\vec{\xi}, -\vec{x}_1, -\vec{x}_2), \quad (\vec{\xi}, -\vec{x}_1, -\vec{x}_2), \quad (-\vec{\xi}, \vec{x}_1, -\vec{x}_2).$$

If we calculate the first and the second fundamental forms of the above focal surfaces

1) for the local surfaces $\vec{p}, \vec{\vec{p}}^*, \vec{\vec{k}}^{**}$, we find,

$$\left.\begin{aligned} E_{II} &= \bar{E}^*_{II} = \bar{E}_I^{**} = \left(\frac{1}{r}\right)_1^2 E \\ F_{II} &= \bar{F}^*_{II} = \bar{F}_I^{**} = \left(\frac{1}{r}\right)_1 \left(\frac{1}{r}\right)_2 \sqrt{EG} \\ G_{II} &= \bar{G}^*_{II} = \bar{G}_I^{**} = \left(\frac{1}{r}\right)_2^2 \frac{r^2 + q^2}{q^2} G, \end{aligned}\right\} \tag{2.1}$$

$$[I]_{II} = [\bar{I}^*]_{II} = [\bar{I}^{**}]_I = \left(\frac{1}{r}\right)_1^2 E\, du^2 + 2 \left(\frac{1}{r}\right)_1 \left(\frac{1}{r}\right)_2 \sqrt{EG}\, du dv$$

$$+ \left(\frac{1}{r}\right)_2^2 \frac{r^2 + q^2}{q^2} G\, dv^2 \tag{2.2}$$

and

$$\left.\begin{aligned} L_{II} &= \bar{L}^*_{II} = \bar{L}_I^{**} = \left(\frac{1}{r}\right)_1 r E \\ M_{II} &= \bar{M}^*_{II} = \bar{M}_I^{**} = 0 \\ N_{II} &= \bar{N}^*_{II} = \bar{N}^{**}_{II} = \left(\frac{1}{r}\right)_2 \frac{r\bar{q}}{q} G, \end{aligned}\right\} \tag{2.3}$$

$$[II]_{II} = [\bar{II}^*]_{II} = [\bar{II}^{**}]_I = r\left[\left(\frac{1}{r}\right)_1 E\, du^2 + \left(\frac{1}{r}\right)_2 \frac{\bar{q}}{q} G\, dv^2\right].$$

From these we may derive the below conclusion:

Conclusion: 2.1. The focal surfaces $\vec{p}, \vec{\vec{p}}^*, \vec{\vec{k}}^{**}$ of the congruences $\vec{y}, \vec{\vec{y}}^*, \vec{\vec{y}}^{**}$ are different positions of the center surface $\vec{r}$ of the base surface $\vec{x}$, in space.

Also, the Gaussian and the mean curvature of these focal surfaces, we find

$$K_{II} = \bar{K}^*_{II} = \bar{K}_I^{**} = \frac{q\bar{q}}{\left(\frac{1}{r}\right)_1 \left(\frac{1}{r}\right)_2} \tag{2.5}$$

and

$$H_{II} = {}^*\overline{H}_{II} = {}_I{}^*\overline{H}^* = \frac{\left(\frac{1}{r}\right)_1 q\bar{q} - \left(\frac{1}{r}\right)_2 (r^2 + q^2)}{2r \left(\frac{1}{r}\right)_1 \left(\frac{1}{r}\right)_2}. \tag{2.6}$$

Since, $r_1 \neq 0$, $r_2 \neq 0$ from (2.1) and (2.3) we derive the conditions $F_{II} = \overline{F}^*_{II} = \overline{F}_I^{**} \neq 0$ and $M_{II} = \bar{M}^*_{II} = \bar{M}_I^{**} = 0$. From these, the following theorem may be stated:

Theorem 2.2. Since the surface $\vec{x}(u, v)$ cannot be a canal surface or at the same time cannot be both Mulür surface and tube–shaped canal surface, the parameter curves $v = \text{const.}$ and $u = \text{const.}$ of the focal surfaces $\vec{p}$, $\vec{\bar{p}}^*$, $\vec{\bar{k}}^{**}$ of the congruences $\vec{y}$, $\vec{\bar{y}}^*$, $\vec{\bar{y}}^{**}$ cannot be the lines of curvature.

Since $q \neq 0$, $\bar{q} \neq 0$, $r_1 \neq 0$, $r_2 \neq 0$ in (2.5) and (2.6), we find $K_{II} = \bar{K}^*_{II} = \bar{K}_I^{**} \neq 0$ and $H_{II} = \overline{H}^*_{II} = \overline{H}_I^{**} \neq 0$.

Therefore the theorem below may be stated:

Theorem 2.3. Since the surface $\vec{x}(u, v)$ cannot be Mulür surface, canal surface or tube–shaped surface, the focal surfaces $\vec{p}$, $\vec{\bar{p}}^*$, $\vec{\bar{k}}^{**}$ of the congruences $\vec{y}$, $\vec{\bar{y}}^*$, $\vec{\bar{y}}^{**}$ respectively, cannot be developable surface, minimal surface.

2) For the local surfaces $\vec{\bar{p}}$, $\vec{\bar{k}}^*$, $\vec{\bar{p}}^{**}$ we find,

$$\left.\begin{aligned}\bar{E}_{II} &= \bar{E}^*_I = \bar{E}^{**}_{II} = \left(\frac{1}{\bar{r}}\right)_1^2 \frac{\bar{r}^2 + \bar{q}^2}{\bar{q}^2} E \\ \bar{F}_{II} &= \bar{F}^*_I = \bar{F}^{**}_{II} = \left(\frac{1}{\bar{r}}\right)_1 \left(\frac{1}{\bar{r}}\right)_2 \sqrt{EG} \\ \bar{G}_{II} &= \bar{G}^*_I = \bar{G}^{**}_{II} = \left(\frac{1}{\bar{r}}\right)_2^2 G,\end{aligned}\right\} \quad (2.7)$$

$$[\bar{I}]_{II} = [\bar{I}^*]_I = [\bar{I}^{**}]_{II} = \left(\frac{1}{\bar{r}}\right)_1^2 \frac{\bar{r}^2 + \bar{q}^2}{\bar{q}^2} E\, du^2 +$$

$$2 \left(\frac{1}{\bar{r}}\right)_1 \left(\frac{1}{\bar{r}}\right)_2 \sqrt{EG}\, dudv + \left(\frac{1}{\bar{r}}\right)_2^2 G\, dv^2 \quad (2.8)$$

and

$$\left.\begin{aligned}\bar{L}_{II} &= \bar{L}^*_I = \bar{L}_{II}^{**} = \left(\frac{1}{\bar{r}}\right)_1 \frac{\bar{r}q}{\bar{q}} E \\ \bar{M}_{II} &= \bar{M}^*_I = \bar{M}_I^{**} = 0 \\ \bar{N}_{II} &= \bar{N}^*_I = \bar{N}^{**}_{II} = \left(\frac{1}{\bar{r}}\right)_2 \bar{r}\, G,\end{aligned}\right\} \quad (2.9)$$

$$[\overline{II}]_{II} = [\overline{II}^*]_I = [\overline{II}^{**}]_{II} = \bar{r} \left[\left(\frac{1}{r}\right)_1 \frac{q}{\bar{q}} E\, du^2 + \left(\frac{1}{\bar{r}}\right)_2 G\, dv\right]. \quad (2.10)$$

From these we may write the below conclusion:

Conclusion 2.4. The focal surfaces $\vec{p}$, $\vec{k}^*$, $\vec{p}^{**}$ of the congruences $\vec{y}$, $\vec{y}^*$, $\vec{y}^{**}$ respectively, are different positions of the center surface $\vec{r}$ of the base surface $\vec{x}$, in space.

Also, the values of K and H for these focal surfaces are found as

$$\bar{K}_{II} = \bar{K}_I^* = \bar{K}_{II}^{**} = \frac{q\bar{q}}{\left(\frac{1}{\bar{r}}\right)_1 \left(\frac{1}{\bar{r}}\right)_2} \quad (2.11)$$

and

$$\overline{H}_{II} = \overline{H}_I{}^* = \overline{H}_{II}{}^{**} = \frac{\left(\frac{1}{\bar{r}}\right)_2 q\bar{q} - \left(\frac{1}{\bar{r}}\right)_1 (\bar{r}^2 + \bar{q}^2)}{2\bar{r}\left(\frac{1}{\bar{r}}\right)_1 \left(\frac{1}{\bar{r}}\right)_2} . \qquad (2.12)$$

Since $\bar{r}_1 \neq 0$, $\bar{r} \neq 0$ from (2.7) and (2.9), we find the conditions $\overline{F}_{II} = \overline{F}_I{}^* = \overline{F}_{II}{}^{**} \neq 0$ and $\overline{M}^*_{II} = \overline{M}_I{}^* = \overline{M}_{II}{}^{**} = 0$.

From these we may write the below theorem:

Theorem 2.5. Since the surface $\vec{x}$ (u, v) cannot be canal surface or tube–shaped surface, the parameter curves v = const. and u = const. of the focal surfaces $\vec{\bar{p}}$, $\vec{\bar{k}}^*$, $\vec{\bar{p}}^{**}$ of the congruences $\vec{\bar{y}}$, $\vec{\bar{y}}^*$, $\vec{\bar{y}}^{**}$ cannot be lines of curvature.

Since $q \neq 0$, $\overline{q} \neq$, 0 $\bar{r}_1 \neq 0$, $\bar{r}_2 \neq 0$ in (2.11) and (2.12), therefore the below theorem may be written.

Theorem 2.6. Since the surface $\vec{x}$ (u, v) cannot be Mulür surface or general cylindric surface, canal surface or tub–shaped surface, the focal surfaces $\vec{\bar{p}}$, $\vec{\bar{k}}^*$, $\vec{\bar{p}}^{**}$ of the congruences $\vec{\bar{y}}$, $\vec{\bar{y}}^*$, $\vec{\bar{y}}^{**}$ respectively cannot be developable surface, minimal surface.

On the other hand to investigate the focal surface $\vec{k}$ and $\vec{\bar{k}}$ belonging to the congruences $\vec{y}$ and $\vec{\bar{y}}$ respectively and coinciding with the center surface of the surfaces $\vec{z}$ and $\vec{\bar{z}}$ but which do not coincide with the center surfaces $\vec{r}$ and $\vec{\bar{r}}$ of the surface $\vec{x}$ (u, v), first, we may write the moving trihedrons connected with the focal point corresponding to ρ_I of $\vec{k}$, connected with the focal point corresponding to $\bar{\rho}_I$ of $\vec{\bar{k}}$ before calculating their first and second fundamental forms.

1) For the focal surface $\vec{k}$, from

$$\vec{k}_1 = -\left(\frac{1}{b}\right)_1 \vec{g}, \quad \vec{k}_2 = \frac{r_2\bar{q}}{r_1 q - q_1 r}\,\vec{x}_1 - \left(\frac{1}{b}\right)_2 \vec{g}$$

and

$$\vec{n}_I = \frac{\vec{k}_1 \wedge \vec{k}_2}{\sqrt{(\vec{k}_1\,\vec{k}_2)}} = \frac{q\vec{x}_2 - r\,\vec{\xi}}{\sqrt{r^2+q^2}} = -\vec{z}_{\bar{1}},$$

$$\vec{n}_I \wedge (-\vec{g}) = \vec{x}_1$$

the trihedron

$$(-\vec{g}, \vec{x}_1, \vec{n}_I) \tag{2.13}$$

is found.

2) For the focal surface $\vec{\vec{k}}$, from

$$\vec{\vec{k}}_1 = -\left(\frac{1}{\bar{\beta}}\right)_1 \vec{\vec{g}} - \frac{\bar{r}_2 q}{\bar{q}_2\bar{r} - \bar{r}_2 q}\,\vec{x}_2, \quad \vec{\vec{k}}_2 = -\left(\frac{1}{\bar{\beta}}\right)_2 \vec{\vec{g}}$$

and

$$\vec{\vec{n}}_I = \frac{\vec{\vec{k}}_2 \wedge \vec{\vec{k}}_1}{\sqrt{(\vec{\vec{k}}_2 \wedge \vec{\vec{k}}_1)}} = \frac{\bar{q}\vec{x}_1 - \bar{r}\vec{\xi}}{\sqrt{\bar{r}^2+\bar{q}^2}} = -\vec{\vec{z}}_{\bar{\bar{2}}},$$

$$\vec{\vec{n}}_I \wedge (-\vec{\vec{g}}) = \vec{x}_2$$

the trihedron is

$$(-\vec{\vec{g}}, \vec{x}_2, \vec{\vec{n}}_I). \tag{2.14}$$

If we calculate the first and the second fundamental forms of the focal surfaces

1) For $\vec{k}$, we find,

$$\left.\begin{aligned} E_I &= \left(\frac{1}{b}\right)_1^2 E \\ F_I &= \left(\frac{1}{b}\right)_1 \left(\frac{1}{b}\right)_2 \sqrt{EG} \\ G_I &= \left[\left(\frac{r_2\bar{q}}{r_1q-q_1r}\right)^2 + \left(\frac{1}{b}\right)_2^2\right] G, \end{aligned}\right\} \tag{2.15}$$

$$[I]_I = \left(\frac{1}{b}\right)_1^2 E\, du^2 + 2\left(\frac{1}{b}\right)_1 \left(\frac{1}{b}\right)_2 \sqrt{EG}\, dudv +$$

$$\left[\left(\frac{r_1\bar{q}}{r_1q-q_1r}\right)^2 + \left(\frac{1}{b}\right)_2^2\right] G\, dv^2 \tag{2.16}$$

and

$$\left.\begin{aligned} L_I &= -\left(\frac{1}{b}\right)_1 \frac{r_1q-q_1r}{r^2+q^2} E \\ M_I &= 0 \\ N_I &= \left(\frac{1}{\bar{b}}\right)_1 \frac{r^2\bar{q}^2}{r_1q-q_1r} G, \end{aligned}\right\} \tag{2.17}$$

$$[II]_I = -\left(\frac{1}{b}\right)_1 \frac{r_1q-q_1r}{r^2+q^2} E\, du^2 + \left(\frac{1}{\bar{b}}\right)_1 \frac{r^2\bar{q}^2}{r_1q-q_1r} G\, dv^2 \tag{2.18}$$

and also,

$$K_I = -\frac{q^2(r_1q-q_1r)^2}{\left(\frac{1}{b}\right)_1 \left(\frac{1}{\bar{b}}\right)_1 r^2(r^2+q^2)^2}, \tag{2.19}$$

$$H_I = \frac{r_1q-q_1r}{2\left(\frac{1}{b}\right)_1 \left(\frac{1}{\bar{b}}\right)_1^2 r^4\bar{q}^2(r^2+q^2)^2} \left\{ \left(\frac{1}{\bar{b}}\right)_1 (r^2+q^2)\, r^2\bar{q}^2 \left[q^2 \left(\frac{1}{b}\right)_1 -\right.\right.$$

$$\left.\left. -r^2 \left(\frac{1}{\bar{b}}\right)_1\right] - \left(\frac{1}{\bar{b}}\right)_2^2 (r_1q-q_1r)\, q^2 \right\}. \tag{2.20}$$

2) For $\vec{\bar{k}}$,

$$\left.\begin{aligned} \bar{E}_I &= \left[\left(\frac{1}{\bar{\beta}}\right)_1^2 + \left(\frac{\bar{r}_2 q}{\bar{q}_2\bar{r} - \bar{r}_2\bar{q}}\right)^2\right] E \\ \bar{F}_I &= \left(\frac{1}{\bar{\beta}}\right)_1 \left(\frac{1}{\bar{\beta}}\right)_2 \sqrt{EG} \\ \bar{G}_I &= \left(\frac{1}{\bar{\beta}}\right)_2^2 G, \end{aligned}\right\} \qquad (2.21)$$

$$[\bar{I}]_I = \left[\left(\frac{1}{\bar{\beta}}\right)_1^2 + \left(\frac{\bar{r}_2 q}{\bar{q}_2\bar{r} - \bar{r}_2\bar{q}}\right)^2\right] E\ du + 2\left(\frac{1}{\bar{\beta}}\right)_1\left(\frac{1}{\bar{\beta}}\right)_2 \sqrt{EG}\,du\,dv + \left(\frac{1}{\bar{\beta}}\right)_2^2 G\ dv^2 \qquad (2.22)$$

and

$$\left.\begin{aligned} \bar{L}_I &= \left(\frac{1}{\beta}\right)_2 \frac{\bar{r}_2 q^2}{\bar{q}_2\bar{r} - \bar{r}_2\bar{q}} E \\ \bar{M}_I &= 0 \\ \bar{N}_I &= \left(\frac{1}{\bar{\beta}}\right)_2 \frac{\bar{r}_2\bar{q} - \bar{q}_2\bar{r}}{\bar{r}^2 + \bar{q}^2} G, \end{aligned}\right\} \qquad (2.23)$$

$$[\bar{II}]_I = \left(\frac{1}{\beta}\right) \frac{\bar{r}^2 q^2}{\bar{q}_2\bar{r} - \bar{r}_2\bar{q}} E\ du^2 + \left(\frac{1}{\bar{\beta}}\right)_2 \frac{\bar{r}_2\bar{q} - \bar{q}_2\bar{r}}{\bar{r}^2 + \bar{q}^2} G\ vd^2 \qquad (2.24)$$

and also,

$$\bar{K}_I = -\frac{\bar{q}^2\,(\bar{q}_2\bar{r} - \bar{q}_2\bar{r})^2}{\left(\frac{1}{\bar{\beta}}\right)_2 \left(\frac{1}{\beta}\right)_2 \bar{r}^2\,(\bar{r}^2 + \bar{q}^2)^2}, \qquad (2.25)$$

$$\bar{H}_I = \frac{\bar{q}_2\,\bar{r} - \bar{r}_2\bar{q}}{2\left(\frac{1}{\bar{\beta}}\right)_2 \left(\frac{1}{\beta}\right)_2^2 (\bar{r}^2 + \bar{q}^2)\,\bar{r}^4\,\bar{q}^2} \qquad (2.26)$$

$$\left\{ \bar{r}^2 q^2 (\bar{r}^2+\bar{q}^2) \left(\frac{1}{\bar{\beta}}\right)_2 \left[\left(\frac{1}{\beta}\right)_2 \bar{r}^2 - \left(\frac{1}{\bar{\beta}}\right)_2 \bar{q}^2\right] - \left(\frac{1}{\bar{\beta}}\right)_2^2 (q_2 \bar{r} - \bar{r}_2 \bar{q})^2 \bar{q}^2 \right\}$$

are written.

1) Since the focal surface $\vec{k}$ coincides with the center surface belonging to the lines of curvature $v = \text{const.}$ of the surface $\vec{z}$, that is $\vec{k} = \vec{r} + \rho \vec{g} = \vec{z} + \frac{1}{b} \vec{n} = \vec{z} - \frac{1}{b} \vec{g}$, we may write,

$$b = b_I = -[(-\vec{g}) \cdot (\vec{n}_I)_1] = -[\vec{n}_I \cdot (-\vec{g})_1] = \frac{r_1 q - q_1 r}{r^2 + q^2} \neq 0, \; (r_1 q - q_1 r \neq 0). \tag{2.27}$$

Since $F_I \neq 0$ ($b_1 \neq 0$, $b_2 \neq 0$) in (2.15) and $M_I = 0$ in (2.17), the below theorem may be written.

Theorem 2.7. Since the surface $\vec{z}$ cannot be canal surface or tube–shaped surface, the parameter curves $v = \text{const.}$ and $u = \text{const.}$ of the focal surface $\vec{k}$ of the congruence $\vec{y}$, cannot be lines of curvature.

From (2.19) and (2.20) $K_I \neq 0$, $H_I \neq 0$ ($q \neq 0$, $r_1 q - q_1 r \neq 0$) are seen. From this the below theorem may be written.

Theorem 2.8. Since the surface $\vec{x}\,(u, v)$ cannot be Mulür surface and the surface which have the lines of curvature $v = \text{const,}$ consisting of plane curves, the focal surface $\vec{k}$ of the congruence $\vec{y}$, cannot be developable surface, minimal surface.

2) Since the focal surface $\vec{\bar{k}}$ coincides with the central surface belonging to the lines of curvature $u = \text{const.}$ on the surface $\vec{\bar{z}}$ that is $\vec{\bar{k}} = \vec{\bar{r}} + \rho \vec{\bar{g}} = \vec{\bar{z}} + \frac{1}{\bar{\beta}} \vec{\bar{n}} = \vec{\bar{z}} - \frac{1}{\bar{\beta}} \vec{\bar{g}}$, we may write

$$\bar{\beta} = \bar{\beta}_I = -[\vec{\bar{n}}_I \cdot (-\vec{\bar{g}})_2] = \frac{\bar{q}_2 \bar{r} - \bar{r}_2 \bar{q}}{\bar{r}^2 + \bar{q}^2} \neq 0, \; (\bar{q}_2 \bar{r} - \bar{r}_2 \bar{q} \neq 0) \tag{2.28}$$

It can be seen that at (2.21), $\overline{F}_I \neq 0$ ($\overline{\beta}_1 \neq 0$, $\overline{\beta}_2 \neq 0$). If we take the condition $\overline{M}_I = 0$ at (2.23) into consideration together with these, we may write the below theorem.

Theorem 2.9. Since the surface $\vec{z}$ cannot be canal surface or tube–shaped surface, the parameter curves v = const. and u = const. on the focal surface $\vec{k}$ of the congruence $\vec{y}$, cannot be lines of curvature.

From (2.25) and (2.26) $\overline{K}_I \neq 0$, $\overline{H}_I \neq 0$ ($\overline{q} \neq 0$, $\overline{q}_2\overline{r} - \overline{r}_2\overline{q} \neq 0$) are seen. From this below theorem may be written.

Theorem 2.10. Since the surface $\vec{x}$ (u, v) cannot be Mulür surface and the surface which have with the lines of curvature u = const. consisting of plane curves, the focal surface $\vec{k}$ of the congruence $\vec{y}$ cannot be developable surface, minimal surface.

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