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SOME GEOMETRIC RESULTS OF GENERALIZED STEREOGRAPHIC PROJECTION

BAKİ KARLIĞA

Department of Mathematics Sciences and Arts Faculty Gazi University 06500 Teknikokullar Ankara TURKEY

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ABSTRACT

We show that geodesics with their causal characters are invariant under stereographic projection of n-dimensional pseudosphere and pseudohyperbolic space with ν-index.

1. INTRODUCTION

Stereographic projection, originally, was discovered by Hiperarch. When Claudius Potelemy was describing the instrument for measuring coordinates of stars on the celestial sphere, so called astrolabe, it was utilized. In 1613, the astrolabe projection was called *stereographic projection* by d'Aguillon in [2]. Stereographic projection was applied to geographic maps and surfaces by Lambert ([12], Euler ([3], [4]), Lagrange ([11]), Gauss ([5]). The comformal disc model for hyperbolic space is obtained by using stereographic projection by Killing ([7], [8]) and Poincare ([15]). In addition to, it is well-known that stereoraphic projection is necessary to determine the inversion of sphere. *Inversion of sphere* was found by L.J. Megnus in 1831. It is also studied in detail by F. Vieta, R. Simson, J. Steiner, L. Kelwin. Stereographic projection has been played very important role in Weiner ([19]), Akutagawa and Nishikawa ([1]), Magid ([13]), Kobayashi ([9]).

The main aim of this paper is to show that geodesics with their causal characters are invariant under streographic projection of n-dimensional pseudosphere with ν-index. The basic definitions and background material required here may be found in Karlığa [6], O'Neil ([14]), Kreyszing ([10]), Schwerdtfeger ([17]), Smart ([18]), Reynolds ([16]).

2. GEODESICS AND THE GENERALIZED STEREOGRAPIC PROJECTION

Definition 2.1. The map

$$\sigma: S_\nu^n(r)\backslash\overline{\Lambda} \to R_\nu^n\backslash H_{\nu-1}^{n-1}(r),$$

$$\sigma(x) = \frac{r}{r - x_{n+1}}(x_1,\ldots,x_n)\ ,\ \overline{\Lambda} = \{x \in S_\nu^n(r) \mid x_n = r\}$$

is called as *generalized stereographic projection* of $S_\nu^n(r)$, $0 \le \nu \le n$ [6].

Theorem 2.1. *The generalized stereographic projection maps the geodesics of $R_\nu^n\backslash H_{\nu-1}^{n-1}(r)$ to the geodesics of $S_\nu^n(r)\backslash\overline{\Lambda}$ by preserving their causal characters.*

Proof. Without no loss of generality we choose the geodesics of $R_\nu^n\backslash H_{\nu-1}^{n-1}(r)$ that pass through the origin. Let α be a unit speed spacelike geodesics of $R_\nu^n\backslash H_{\nu-1}^{n-1}(r)$. Then we can take $\alpha(t) = tv$ and $\langle v,v\rangle = 1$. By Theorem 2.1 of [6], we find

$$\beta(t) = \sigma^{-1} \circ \alpha(t) = \frac{r}{r^2 + \langle\alpha(t),\alpha(t)\rangle}(2rv_1t,\ldots,2rv_nt\ ,\ \langle\alpha(t),\alpha(t)\rangle - r^2).$$

$$\beta(t) = \frac{2r^2t}{r^2 + t^2}(v_1\ ,\ \ldots\ ,\ v_n\ ,\ 0) + \frac{r(t^2 - r^2)}{r^2 + t^2}(0\ ,\ \ldots\ ,\ 0\ ,\ 1).$$

or

$$\beta(t) = \frac{2r^2t}{r^2 + t^2}E_1 + \frac{r(t^2 - r^2)}{r^2 + t^2}E_2.$$

where $E_1 = (v_1,\ldots,v_n,0)$ and $E_2 = (0,\ldots,0,1)$ $\langle E_i,E_j\rangle = \delta_{ij}$, $1 \le i, j \le 2$. By Theorem 2.2 of [6], we get

$$\langle\dot\beta(t),\dot\beta(t)\rangle = \frac{r^2}{(r^2 + t^2)^2}$$

If we choose β as the arclength parametrization then, after routine calculations we find

$$\beta(s) = f(s)E_1 + g(s)E_2$$

$$\dot\beta(s) = \dot f(s)E_1 + \dot g(s)E_2$$

$$\ddot\beta(s) = \ddot f(s)E_1 + \ddot g(s)E_2$$

Hence $\beta(s), \dot{\beta}(s)$ and $\ddot{\beta}(s)$ belong to the subspace W which is spanned by $\{E_1,E_2\}$. Since $\langle\beta(s),\beta(s)\rangle = r^2$ we have the following equations

$$\langle\dot{\beta}(s)\ ,\ \beta(s)\rangle = 0 \tag{1}$$

$$\langle\ddot{\beta}(s)\ ,\ \beta(s)\rangle = -1 \tag{2}$$

and

$$\langle\ddot{\beta}(s)\ ,\ \dot{\beta}(s)\rangle = 0 \tag{3}$$

by (1) we set that $\{\beta(s)\ ,\ \dot{\beta}(s)\}$ is ortoghonal and so, $Sp\{\beta(s)\ ,\ \dot{\beta}(s)\}$ $Sp\{E_1,E_2\}$. This implies that there exist functions λ_1, λ_2 such that

$$\ddot{\beta}(s) = \lambda_1(s)\beta(s) + \lambda_2(s)\dot{\beta}(s)$$

By (1),(2) and (3), we find $\lambda_1(s) = \frac{-1}{r^2}$ and $\lambda_2(s) = 0$ and so $\ddot{\beta}(s) = \frac{-1}{r^2}\ \beta(s)$. This implies that β is a spacelike geodesic of $S_v^n(r)\backslash\overline{\Lambda}$ as required.

If we take α as a timelike geodesic of $R_v^n\backslash H_{v-1}^{n-1}(r)$ which pass through the origin then, by following the above arguments and steps it is not difficult to show that σ^{-1} maps α to a timelike geodesic of $S_v^n(r)\backslash\overline{\Lambda}$.

On the other hand, when α is a null geodecis of $R_v^n\backslash H_{v-1}^{n-1}(r)$ which pass through the origin; routine calculations show that σ^{-1} maps α to a null geodesic of $S_v^n(r)\backslash\overline{\Lambda}$ ❑

Theorem 2.2. The generalized stereographic projection maps the geodesics $R_v^n\backslash S_{v-1}^{n-1}(r)$ of to the geodesics of $H_{n-v}^n(r)\backslash\overline{\Lambda}$ by preserving their causal characters.

Proof. It follows from Theorem 4.2 of [6] and similar arguments of Theorem 2.1 ❑

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