Commun. Fac. Sci. Univ. Ank. Series A1
V. 45. pp. 55-60 (1996)

ON THE SUBRINGS OF THE RING OF ANALYTIC FUNCTIONS AND CONFORMALLY EQUIVALENCE

EKREM KADIOĞLU* and MUSTAFA BAYRAKTAR**

* *Department of Mathematics, Atatürk University, Faculty of Sciences and Arts, 25240 Erzurum-TURKEY*
***Department of Mathematics, Uludağ University, Faculty of Sciences and Arts, Bursa-TURKEY*

(Received Nov. 6, 1995; Revised Apr. 12, 1996; Accepted June 13, 1996)

ABSTRACT

We consider the discrete sets $D_i \subset G_i$ of the regions G_i in the complex plane $\mathcal{C}$ and study the subrings of the rings of analytic functions $A(G_i)$ corresponding to these discrete sets D_i (i=1,2). Furthermore, we prove that the sets of zeros of the functions which map onto each other under the $\mathcal{C}$-isomorphism Φ: $A(G_1) \to A(G_2)$, are also mapped onto each other by a conformal mapping φ: $G_2 \to G_1$, where $\Phi(f)=f\circ\varphi$.

1. INTRODUCTION

Helmer had shown that finitely generated ideal in the ring of entire functions was a principle ideal [2]. After this work, some other authors have started to work on the rings of analytic functions, and conformal equivalence was characterized algebraically.

Let G_1 and G_2 be two domains in the complex plane, and let $A(G_1)$ and $A(G_2)$ be the rings of analytic functions of G_1 and G_2 respectively. If these exists a $\mathcal{C}$-isomorphism between $A(G_1)$ and $A(G_2)$, then G_1 and G_2 are conformally equivalent [1]. The problem was generalized to open Riemann surfaces G_1 and G_2 [4]. It was shown that two domains G_1 and G_2 in the complex plane were conformally equivalent if the rings $B(G_1)$ and $B(G_2)$ of all bounded analytic functions defined on them were algebraically $\mathcal{C}$-isomorphic [3]. When we discuss the rings $B(G_i)$ (i=1,2), it is always assumed that G_i is bounded and has the following property: for any $z \in \partial G_i$, boundary of G_i, there exists a function $f \in B(G_i)$ for which z is an unremovable singularity. It is proved that if there is a

$\mathcal{C}$-isomorphism between $A(G_1)$ and $A(G_2)$, then the sets G_1 and G_2 are conformally equivalence [5].

Definition 1.1. Let S be any non-empty subset of the complex plane $\mathcal{C}$ and f: $S \to \mathcal{C}$ be a function. If f is an analytic function in the domain which contains S, f is called an analytic function is S.

Let G be any non-empty subset of $\mathcal{C}$ and A(G) be the set of single-valued analytic functions on G. The set A(G) is a ring with respect to two binary operations which are defined by $(f+g)(z) = f(z)+g(z)$ and $(fg)(z) = f(z)g(z)$.

Definition 1.2. Let G_1 and G_2 be two non-empty subsets of $\mathcal{C}$. If the mapping φ: $G_1 \to G_2$ is analytic and bijective, then φ is called a conformal mapping from G_1 to G_2. In this case, G_1 and G_2 are called conformally equivalent.

In this work, an analytic function means that it is a single-valued analytic function.

Let $a \in G$ be an arbitrary but fixed point. The set $M_a = \{f \in A(G): f(a) = 0\}$ is a maximal ideal which is generated by z-a from A(G). M_a is called a fixed maximal ideal of A(G), and all other maximal ideals of A(G) are called free maximal ideal.

Definition 1.3. Let G be a region (or a set) in the complex plane and $D \subset G$. If D has no limit point in G, then D is called a discrete subset of G.

Theorem 1.4. Let G_1 and G_2 be two subsets of $\mathcal{C}$, and Φ be an $\mathcal{C}$-isomorphism from $A(G_1)$ onto $A(G_2)$. Then Φ induces a mapping φ: $G_2 \to G_1$, defined by $\Phi(f) = f o \varphi$, and φ is a conformal mapping of G_2 onto G_1 [5].

Theorem 1.5. Let G_1 and G_2 be two subsets of $\mathcal{C}$, and φ: $G_2 \to G_1$ be a conformal mapping. Then the mapping Φ defined by $\Phi(f) = f o \varphi$ is a $\mathcal{C}$-isomorphism from $A(G_1)$ onto $A(G_2)$ [5].

2. SUBRINGS OF THE RING A(G)

In this section, subrings of the ring A(G) corresponding to the discrete sets will be investigated.

Theorem 2.1. Let

$$A_{D_1}(G_1) = \{f \in A(G_1): f(u) = \text{constant, for all } u \in D_1\}$$

and

$$A_{D_2}(G_2) = \{g \in A(G_2): g(v) = \text{constant, for all } v \in D_2\}$$

where D_1 and D_2 are discrete subsets of G_1 and G_2, respectively. Let $\varphi: G_2 \to G_1$ be a bijective analytic mapping. If $\varphi(D_2) = D_1$, then $\Phi: A_{D_1}(G_1) \to A_{D_2}(G_2)$, $\Phi(f) = f o \varphi$ is a $\mathcal{C}$-isomorphism.

Proof: It is easily seen that $\Phi(f_1 f_2) = \Phi(f_1)\Phi(f_2)$ and $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$. Hence Φ is a homomorphism. Since g is an element of $A_{D_2}(G_2)$, $go\varphi^{-1} \in A_{D_1}(G_1)$. If $\varphi^{-1}(c) \in D_2$, $c \in D_1$, then $(go\varphi^{-1})(c) = g(\varphi^{-1}(c)) =$ constant. Therefore, for all $g \in A_{D_2}(G_2)$, there exists $go\varphi^{-1} \in A_{D_1}(G_1)$ such that $\Phi(go\varphi^{-1}) = (go\varphi^{-1})o\varphi = g$. Hence Φ is onto. On the other hand, since

$$\Phi(f_1) = \Phi(f_2) \Rightarrow f_1 o\varphi = f_2 o\varphi \Rightarrow f_1 = f_2$$

Φ is one to one. It is obvious that the constants are invariant under the isomorphism Φ. Hence Φ is a $\mathcal{C}$-isomorphism.

The following theorem gives us a relation between the ring A(G) and the subring $A_D(G)$.

Theorem 2.2. If a discrete set $D \subset G$ contains only one element, i.e., $D = \{a\}$, then $A_D(G) = A(G)$.

Proof: From definition of $A_D(G)$, we have $A_D(G) \subset A(G)$. On the other hand, suppose that $f \in A(G)$. In this case, since $f(a) =$ constant, f is an element of $A_D(G)$. Then $A(G) \subset A_D(G)$ holds. Therefore, $A(G) = A_D(G)$.

By this theorem, we can say that we will be able to work on the subring $A_D(G)$ instead of the ring A(G). If D has only one element, we can take the set $A_D(G)$ for A(G).

Corollary 2.3. Let D_1 and D_2 be two discrete subsets of G. If $D_1 \subset D_2$

$$A_{D_1}(G) = \{f \in A(G): f(u) = \text{constant, for all } u \in D_1\}$$

and

$$A_{D_2}(G) = \{g \in A(G): g(v) = \text{constant, for all } v \in D_2\}$$

then $A_{D_2}(G) \subset A_{D_1}(G)$.

Proof: Let $f \in A_{D_2}(G)$. Then for all $v \in D_2$, $f(v) = $ constant. Since $D_1 \subset D_2$ and $f(u) = $ constant for all $u \in D_1$, $f \in A_{D_1}(G)$. Hence, we have $A_{D_2}(G) \subset A_{D_1}(G)$.

Let $D_f \subset G$ be a set of zeros of $f \in A(G)$. Now we can give the following theorem on maximal ideals of $A_{D_f}(G)$.

Theorem 2.4. Let $f \in A(G)$ and D_f be a finite set. Furthermore suppose that

$$A_{D_f}(G) = \{g \in A(G): g(u) = \text{constant, for all } u \in D_f\}$$

Then

$$\{g(z) \prod_{u \in D_f} (z-u): g \in A(G)\}$$

is a maximal ideal of $A_{D_f}(G)$.

Proof. It is ealisy shown that $J = \{g(z) \prod_{u \in D_f} (z-u): g \in A(G)\}$ is an ideal of $A_{D_f}(G)$. Now, we will show that this ideal is maximal. Let us consider the mapping $\psi_u: A_{D_f}(G) \to \mathcal{C}$, $\psi_u(g) = g(u)$, $u \in D_f$. It is clear that the mapping ψ_u is a homomorphism. There exists $h = f^n + c \in A_{D_f}(G)$ such that $\psi_u(h) = h(u) = c$, $(c \in \mathcal{C})$, hence ψ_u is onto. At the same time

$$\text{Ker}\psi_u = \{h \in A_{D_f}(G): \psi_u(h) = h(u) = 0\}.$$

Since $h(u)=0$ for all $u \in D_f$ and $h \in A_{D_f}(G)$, we get $h(z)=g(z) \prod_{u \in D_f} (z-u)$, where $g \in A(G)$. Hence $\text{Ker}\psi_u = J$. According to the first isomorphism theorem

$$A_{D_f}(G)/J \cong \mathcal{C}.$$

Hence, J is a maximal ideal.

$M_a = \{g\in A(G): g(a) = 0\}$ is a maximal ideal of A(G), where $a\in G$. In Theorem 2.4, taking a discrete set instead of a, this result is generalized for maximal ideals.

Theorem 2.5. Let D_{f_1} and D_{f_2} be sets of zeros of $f_1\in A(G_1)$ and $f_2\in A(G_2)$, respectively. Moreover, suppose that the mapping Φ: $A(G_1) \to A(G_2)$ defined by $\Phi(f) = f o\varphi$ is a $\mathscr{C}$-isomorphism. If $\Phi(f_1) = f_2$, then $\varphi(D_{f_2}) = D_{f_1}$.

Proof. From Theorem 2.4, G_2 and G_1 are conformally equivalent, i.e., there exists a mapping φ: $G_2\to G_1$ which is analytic and bijective. From the hypothesis $f_2 = f_1 o\varphi$. If $a\in D_{f_2}$, then

$$0 = f_2(a) = (f_1 o\varphi)(a) = f_1(\varphi(a)).$$

Thus, $\varphi(a)\in D_{f_1}$. Since a is an arbitrary element of D_{f_2}, we have that $\varphi(D_{f_2})\subset D_{f_1}$. On the other hand, if $d\in D_{f_1}$, then there exists $c\in G_2$ such that $\varphi(c) = d$. Hence

$$0 = f_1(d) = f_1(\varphi(c)) = (f_1 o\varphi)(c) = f_2(c)$$

and $c\in D_{f_2}$. Thus $D_{f_1}\subset\varphi(D_{f_2})$. Then the result follows.

We can give the following theorem as a corollary of Theorem 2.1 and 2.5.

Theorem 2.6. If F: $A(G_1)\to A(G_2)$ is a $\mathscr{C}$-isomorphism and $\Phi(f_1) = f_2$, then

$$A_{D_{f_1}}(G_1) \cong A_{D_{f_2}}(G_2).$$

Proof. From the hypothesis and Theorem 2.4, G_2 and G_1 are conformally equivalent, i.e., there exists a mapping φ: $G_2\to G_1$ which is analytic and bijective. According to Theorem 2.5. $\varphi(D_{f_2}) = D_{f_1}$. From Theorem 2.1, we have

$$A_{D_{f_1}}(G_1) \cong A_{D_{f_2}}(G_2).$$

REFERENCES

[1] BERS, L., On rings of analytic functions, Bull. Amer. Math. Soc. 54 (1948), 311-315.

[2] HELMER, O., Divisibility properties of integral functions, Duke Math. J., 6 (1940), 345-356.

[3] KAKUTANI, S., Rings of analytic functions, Proc. Michigan Conference on Functions of a Complex Variable. (1955), 71-74.

[4] RUDIN, W., An algebraic characterization of conformal equivalence, Bul. Amer. Math. Soc. 61 (1955), 543.

[5] SU, L.P., Rings of analytic functions on any subset of the complex plane, Pacific J. Math. 42 (1972), 535-538.