Commun. Fac. Sci. Univ. Ank. Series A_1
V. 44. pp. 81-84 (1995)

A NOTE ON ZHONA'S INVERSE THEOREM

DURMUŞ BOZKURT

Selçuk University Faculty of Arts and Sciences Dept. of Maths., 42079, *Konya–TURKEY*

(Received Aug, 3, 1993; Accepted Dec. 7, 1995)

ABSTRACT

In this paper we have shown that Zhong's inverse theorem is also true for any square matrix. We gave three definitions and obtained two theorems using these definitions.

1. INTRODUCTION

Xu Zhong gave the following theorem and lemma for the inverses of quasi–Hessenberg matrices in [1]:

Theorem 1. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \gamma^T \\ \alpha & \beta & T \\ a_{n1} & a_{n2} & \delta^T \end{bmatrix} \tag{1}$$

be a quasi–Hessenberg matrix such that

$b_2.b_3\ldots b_n \neq 0,\ \alpha = (a_{21}, a_{31}, \ldots, a_{n-1,1})^T,\ \beta = (a_{22}, a_{32}, \ldots, a_{n-1,2})^T$

$\gamma^T = (0, \ldots, 0, b_n),\ \delta^T = (a_{n3}, a_{n4}, \ldots, a_{nn})$

and

$$T = \begin{bmatrix} b_2 & & \\ a_{33} & b_3 & \\ \cdot & \cdot & \cdot \\ a_{n-1,2} & a_{n-1,4} & b_{n-1} \end{bmatrix} \tag{2}$$

If A is nonsingular, then

$$A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & T^{-1} & & \vdots \\ 0 & & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ u & v \end{bmatrix} D^{-1} \begin{bmatrix} 1 & x^T & 0 \\ 0 & y^T & 1 \end{bmatrix} \tag{3}$$

Lemma 1. Let the matrix A as in (1) satisfy $b_2 \ldots b_n \neq 0$. Then A is nonsingular if and only if

$$\det D \neq 0$$

where

$$D = \begin{bmatrix} a_{11} + x^T\alpha & b_1 + x^T\beta \\ a_{n1} + y^T\alpha & a_{n2} + y^T\beta \end{bmatrix} = \begin{bmatrix} a_{11} + \gamma^T u & b_1 + \gamma^T v \\ a_{n1} + \delta^T u & a_{n2} + \delta^T v \end{bmatrix} \qquad (4)$$

such that

$$u = -T^{-1}\alpha, \; v = -T^{-1}\beta, \; x^T = -\gamma^T T^{-1}, \; y^T = -\delta^T T^{-1}. \qquad (5)$$

Proof: From the relation

$$\begin{bmatrix} 1 & x^T & 0 \\ 0 & I_{n-2} & 0 \\ 0 & y^T & 1 \end{bmatrix} \begin{bmatrix} a_{11} & b_1 & \gamma^T \\ \alpha & \beta & T \\ a_{n1} & a_{n2} & \delta^T \end{bmatrix} \begin{bmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ u & v & I_{n-2} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ & 0 & T \\ d_{21} & d_{22} & 0 \end{bmatrix} \qquad (6)$$

we get

$$\det A = (-1)^{n-2} \det T . \det D.$$

where d_{ij} are the elements of the matrix D in (4) for $i, j = 1, 2$. Thus

$$\det A \neq 0 \Leftrightarrow \det D \neq 0.$$

If follows from (6) that

$$A^{-1} = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & 0 \\ u & v & I_{n-2} \end{bmatrix} \begin{bmatrix} d_{11} & & d_{12} & 0 \\ & 0 & & T \\ d_{21} & & d_{22} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & x^T & 0 \\ 0 & I_{n-2} & 0 \\ 0 & y^T & 1 \end{bmatrix}. \qquad (7)$$

2. INVERSE OF ANY SQUARE MATRIX

Theorem 2. Let A be $n \times n$ square matrix. Assume that the matrix A is partitioned in (1). If the matrix A is nonsingular, then the inverse A^{-1} of A satisfy the formulea (3).

Proof: First of all, we must show that the matrix A satisfy the formulea (6). If D is selected as in (4), then the formulea (6) is easily written. Therefore, Lemma 1 is true for the matrix A. Hence if the matrix A is nonsingular, then the matrix D is nonsingular. If the inverse of two side of the formulea (6) is calculated, then it is obtained the formulea (7). Let

$$D^{-1} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

It is easily verified that

$$\begin{bmatrix} d_{11} & & d_{12} & 0 \\ & 0 & & T \\ d_{21} & & d_{22} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} m_{11} & & m_{12} \\ & 0 & \\ m_{21} & & m_{22} \\ & T^{-1} & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 0 \\ 0 & T^{-1} & 0 \end{bmatrix} + \begin{bmatrix} m_{11} & & m_{12} \\ & 0 & \\ m_{21} & & m_{22} \\ 0 & 0 & 0 \end{bmatrix}.$$

Substituting this formulea in (7) gives (3). Thus the proof of Theorem 2 is completed.

Definition 1. Let the matrix A be a square matrix. The matrix A is called an lower almost Hessenberg matrix (transposed matrix A^T is an upper almost Hessenberg matrix) if the elements a_{ij} of A are zero except for the unique a_{rs} where $j - i \geq 2$, $1 \leq r \leq n-2$ and $r + 2 \leq s \leq n$.

Definition 2. Let the matrix A be a square matrix. The matrix A is called an almost tridiagonal matrix if the elements a_{ij} of A are zero except for the unique a_{rs} and a_{sr} where $|i - j| > 1$, $1 \leq r \leq n-2$ and $r + 2 \leq s \leq n$.

Definition 3. Let the matrix A be a square matrix. The matrix A is called an almost lower (almost upper) triangular matrix if the elements a_{ij} (a_{ji}) of A are zero except for the unique where $j - i \geq 1$, $1 \leq r \leq n-1$ and $r + 1 \leq s \leq n$.

Theorem 3. Suppose that the matrix A is an almost lower Hessenberg matrix such that A is nonsingular and the element $a_{i,\,i+2}$ of A is nonzero for $1 \leq i \leq n-2$. Then the minors $A\begin{pmatrix} i & , i+1 \\ i+1, & i+2 \end{pmatrix}$ of the matrix A are nonzero if and only if the inverse A^{-1} of A is written as (3) for $i = 2, 3, \ldots, n-2$.

Proof: We must show that there exist the inverses T^{-1} and D^{-1} of T and D respectively. Let the minors $A\begin{pmatrix} i & , i+1 \\ i+1, & i+2 \end{pmatrix}$ be nonzero for $i = 2, 3, \ldots, n-2$. Then

$$T = \begin{bmatrix} a_{23} & 0 & \ldots 0 & 0 & 0 & 0 & \ldots 0 \\ a_{33} & a_{34} & \ldots 0 & 0 & 0 & 0 & \ldots 0 \\ \cdot & \cdot & \ldots \cdot & \cdot & \cdot & \cdot & \ldots \cdot \\ a_{i3} & a_{i4} & \ldots a_{ii} & a_{i,\,i+1} & a_{i,\,i+2} & 0 & \ldots 0 \\ a_{i+1,\,3} & a_{i+1,\,4} & \ldots a_{i+1,\,i} & a_{i+1,\,i+1} & a_{i+1,\,i+2} & 0 & \ldots 0 \\ \cdot & \cdot & \ldots \cdot & \cdot & \cdot & \cdot & \ldots \cdot \\ a_{n-1,\,3} & a_{n-1,\,4} & \ldots a_{n-1,\,i} & a_{n-1,\,i+1} & a_{n-1,\,i+3} & a_{n-1,\,i+3} & \ldots a_{n-1,\,n} \end{bmatrix}$$

Since the matrix T is an almost lower triangular matrix, we have

$$\det T = \prod_{i=2}^{n-2} A\begin{pmatrix} i & , i+1 \\ i+1, & i+2 \end{pmatrix}.$$

Since $A\begin{pmatrix} i & , i+1 \\ i+2, & i+2 \end{pmatrix}$ are nonzero from hypothesis, $\det T \neq 0$. Hence there exist T^{-1}. Therefore A is nonsingular. Hence there exist D^{-1} form Lemma 1. Thus A^{-1} is writtten os in the form (3).

Let A^{-1} be written as in the form (3). Then, there exist D^{-1} and T^{-1}. Since there exist T^{-1}, $\det T \neq 0$ and since

$$\det T = \prod_{i=2}^{n-2} A\begin{pmatrix} i & , i+1 \\ i+1, & i+2 \end{pmatrix}$$

and $a_{i,\,i+2} \neq 0$, the minors $A\begin{pmatrix} i & , i+1 \\ i+1, & i+2 \end{pmatrix}$ of A are nonzero. The proof of Theorem 3 is completed.

Corollary 1. By taking the transpose, we see that similar theorem holds for an almost upper Hessenberg matrix.

Theorem 4. Let the matrix A be an almost tridiagonal matrix such that $a_{i,\,i+2} \neq 0$ for $1 \leq i \leq n-2$. the minors $A\begin{pmatrix} i & , i+1 \\ i+1, & i+2 \end{pmatrix}$ of A are nonzero if and only if A^{-1} is written as in (3) for $i = 2, \ldots, n$.

Proof is obvious from the proof of Theorem 3.

REFERENCES

[1] X. ZHONG., An Algorithm for the Inversion of Quasi–Hessenberg Matrices, Linear Algebra and its Applications 144: 39–47 (1991).