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ON HELICES OF A LORENTZIAN MANIFOLD

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ABSTRACT

T. Ikawa obtained in [1] the following differential equation

$$D_xD_xD_xX - KD_xX = 0 \ , \ K = k_1^2 - k_2^2$$

for the circular helix which corresponds to the case that the curvatures k_1 and k_2 of a time-like curve α on the Lorentzian manifold M_1 are constants.

In this paper, T. Ikawa's result is generalized to the case of general helix, i.e. k_1 and k_2 are non-constant functions of t, but $\frac{k_1}{k_2}$ is constant.

1. PRELIMINARIES

$\mathbb{R}^n$ with the metric tensor

$$\langle V_p,W_p\rangle = -\sum_{j=1}^{i} V_jW_j + \sum_{k=i+1}^{n} V_kW_k \ , \ V_p,W_p \in \mathbb{R}^n$$

is called semi-Euclidean space and is denoted by $\mathbb{R}_i^n$ where i is called the index of the metric [2].

Let M be an n-dimensional smooth manifold equipped with a metric $\langle \ , \ \rangle$ which is a symmetric non-degenerate (0,2)-tensor field on M with constant index.

A tangent space $T_p(M)$ at the point $p \in M$ is furnished with the canonical inner product. If the index of the metric $\langle \ , \ \rangle$ is i, then we call M and indefinite-Riemannian manifold of index i and denote it by M_i. If $\langle \ , \ \rangle$ is positive definite, then M is a Riemannian manifold. Especially if i=1, then M is called a Lorentzian manifold. A tangent vector X of M_i is said to be space like if $\langle X,X\rangle > 0$, time-like if $\langle X,X\rangle < 0$ and null if $\langle X,X\rangle = 0$ and $X \neq 0$.

Let $X_1,...,X_i,X_{i+1},...,X_n$ be tanget vectors of M_i, n=dim M. Assume that they satisfy $\langle X_A,X_B\rangle = \varepsilon_A\delta_{AB}$ where $\varepsilon_A = \langle X_A,X_A\rangle = +1$ (resp. -1) for A = 1, 2, ... , n. If each X_A is space-like (resp. time-like) then $\{X_1,...,X_n\}$ is called an orthonormal basis of M_i [2].

2. CURVES

A curve in an indefinite-Riemannian manifold M_i is a smooth mapping

$$\alpha: I \to M_i$$

where I is an open interval in the real line $\mathbb{R}$. The interval I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $t \in I$ is

$$\alpha'(t) = \left.\frac{d\alpha(u)}{du}\right|_t$$

A curve α is said to be regular if $\alpha'(t)$ does not vanish for all t in I.

A curve α in an indefinite-Riemannian manifold M_i is said to be space-like if its velocity vectors α' are space-like for all $t \in I$; similarly for time-like and null. If α is a space-like or time-like curve, we can reparametrize it such that $\langle\alpha'(t),\alpha'(t)\rangle = \varepsilon$ (where $\varepsilon=+1$ if α is space-like and $\varepsilon=-1$ if α is time-like respectively). In this case α is said to be unit speed or it has arc length parametrization. Here and in the sequal, we assume that the space-like or time-like curve α has an arc length parametrization.

We define here a circle and circular helix in an indefinite-Riemannian manifold M_i (cf[3], [4]). Let α be a time-like curve in M_i. By k_j, we denote the j-th curvature of α. If k_j vanishes for $j > 2$ an the principal vector field Y and binormal vector field Z are space-like, then we have the following Frenet formulas along α

$$\left.\begin{aligned} \alpha'(t) &=: X \\ D_x X &= k_1 Y \\ D_x Y &= k_1 X + k_2 Z \\ D_x Z &= -k_2 Y \end{aligned}\right\} \qquad (2.1)$$

where D denotes the covariant differentiation in M_i. A curve α is called a circle if $k_2 \equiv 0$ and k_1 = constant > 0, for all $t \in I$.

If both k_1 and k_2 are positive constants along α, then α is called a circular helix [1].

Definition 2.1: A general helix is a regular curve α such that for some fixed unit vector U, $\langle T,U\rangle$ is constant. U is called the axis of a helix [5].

Corollary: (Lancert, 1802). A unit speed curve α with $k_2 \neq 0$ is a general helix if and only if there is a constant c such that $k_1(t) = ck_2(t)$ for all $t \in I$ [5].

3. CIRCLES

Let α be a regular time-like curve in a Lorentzian manifold M_1. In this section, we assume that α is a circle, that is, α satisfies

$$\left.\begin{aligned} \alpha'(t) &= X \\ D_x X &= k_1(t)Y \\ D_x Y &= k_1(t)X \end{aligned}\right\} \tag{3.1}$$

for any $t \in I$, where Y is a space-like vector field and k_1 a positive constant function of the parameter t.

Lemma 3.1: Let α be a time-like curve in a Lorentzian manifold M_1. If α is a circle, then the velocity vector field X or α satisfies

$$D_X D_X X - \langle D_X X, D_X X\rangle X = 0 \tag{3.2}$$

conversely, if the velocity vector field of a time-like curve α satisfies (3.2), then α is either a geodesic or a circle [1].

4. HELICES

Next we consider general helices in a Lorentzian manifold M_1. Then we have

$$\left.\begin{aligned} \alpha'(t) &= X \\ D_x X &= k_1 Y \\ D_x Y &= k_1 X + k_2 Z \\ D_x Z &= -k_2 Y \end{aligned}\right\} \tag{4.1}$$

for any $t \in I$, where Y, Z are space-like vector fields and k_1,k_2 are the functions of the parameter t.

Theorem 4.1: A unit speed curve α on M_1 is a general helix if and only if

$$D_xD_xD_xX - \overline{K}D_xX = 3k_1'(t)D_xY \tag{4.2}$$

where

$$\overline{K} = \frac{k_1''(t)}{k_1'(t)} + k_1^2(t) - k_2^2(t) \tag{4.3}$$

Proof: Suppose that α is a general helix. Then, from (4.1), we have,

$$\begin{aligned} D_xD_xX &= D_x(k_1Y) \\ &= k_1'Y + k_1D_xY \\ &= k_1^2X + k_1'Y + k_1k_2Z \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} D_xD_xD_xX &= 3k_1'k_1X + (k_1'' - k_1k_2^2)Y \\ &\quad + (k_1'k_2 + (k_1k_2)')Z + k_1^2D_xX \end{aligned} \tag{4.5}$$

Now, since α is a general helix, we have

$$\frac{k_1}{k_2} = \text{constant}$$

and this upon the derivation gives rise to

$$k_1'k_2 = k_1k_2' \ .$$

If we substitute the values

$$Y = \frac{1}{k_1(t)} D_xX \tag{4.6}$$

and

$$(k_1(t)\ k_2(t))' = 2k_1'(t)\ k_2(t)\ ,$$

in (4.5) we obtain

$$\begin{aligned} D_xD_xD_xX &= \left(\frac{k_1''(t)}{k_1(t)} + k_1^2(t) - k_2^2(t)\right) D_xX + 3k_1'(t)\ (k_1(t)X + k_2(t)Z) \\ &= \left(\frac{k_1''(t)}{k_1(t)} + k_1^2(t) - k_2^2(t)\right) D_xX + 3k_1'(t)D_xY. \end{aligned}$$

Hence we have (4.2).

Conversely let us assume that (4.2) holds. We show that the curve α is a general helix. Differentiating covariantly (4.6) we obtain

$$D_xY = -\frac{k_1'(t)}{k_1^2(t)}D_xX + \frac{1}{k_1(t)}D_xD_xX$$

and so,

$$D_xD_xY = \left(-\frac{k_1'(t)}{k_1^2(t)}\right)' D_xX - \frac{k_1'(t)}{k_1^2(t)}D_xD_xX$$

$$-\frac{k_1'(t)}{k_1^2(t)}D_xD_xX + \frac{1}{k_1(t)}D_xD_xD_xX \tag{4.7}$$

if we use (4.2) in (4.7), we get

$$D_xD_xY = \left\{\left(-\frac{k_1'(t)}{k_1^2(t)}\right)' + \frac{\overline{K}}{k_1(t)}\right\}D_xX$$

$$-\frac{2k_1'(t)}{k_1^2(t)}D_xD_xX + \frac{3k_1'(t)}{k_1(t)}D_xY\ .$$

Substituting (4.4) and (4.1) in this last equality we obtain

$$D_xD_xY = \left\{\left(-\frac{k_1'(t)}{k_1^2(t)}\right)' + \frac{\overline{K}}{k_1(t)}\right\}D_xX$$

$$-\frac{2k'^2_1(t)}{k_1^2(t)}Y + k_1'(t)X + \frac{k_1'(t)k_2(t)}{k_1(t)}Z \tag{4.8}$$

On the other hand substituting the equality

$$D_xD_xY = k_1'(t)X - k_2^2(t)Y + k_2'(t)Z + k_1(t)D_xX$$

in (4.8) we obtain

$$k_2'(t) = \frac{k_1'(t)k_2(t)}{k_1(t)}$$

and so

$$\frac{k_2'(t)}{k_2(t)} = \frac{k_1'(t)}{k_1(t)}\ .$$

Integrating this we get

$$\frac{k_1(t)}{k_2(t)} = \text{constant}.$$

Thus α is a general helix. Hence the proof is done.

We note that in the special case when α is a circular helix, our theorem coincides with the result of T. Ikawa [1].

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