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QUASI-COMPLEMENTARITY PROBLEMS IN HILBERT SPACES

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ABSTRACT

In this paper, we use quasivariational inequailty and fixed point techniques to derive an existence and uniqueness result for a class of quasi-complementarity problems. Further, we give an iterative algorithm for this class of problems and discuss its convergence criteria.

1. INTRODUCTION

Complementarity theory, introduced and studied by Lemke [7] and Cottle and Dantzig [2] in the early sixtees, has enjoyed vigorous growth for the last two decades. This theory has been extended and generalized in various directions to study a wide class of problems arising in optimization and control, operation research, fluid flow through porous media, mechanics, economics and transportation equilibrium, management sciences, etc. Among these generalizations of the complementarity problem, the quasi-complementarity problem considered and studied by Pang [13, 14] and Noor [8-12] is important and useful generalization. Related to the complementarity problem, there is also a variational inequality problem. Karamardian [5] has shown that if the underlying set in both these problems is a convex cone, then complementarity problem and variational inequality problem are equivalent. Pang [13] proved that the same relation is true for the quasi-complementarity problem and the quasivariational inequality problem. This equivalence has been used quite effectively in suggesting unified and general algorithms for solving complementarity and quasi-complementarity problems.

Let H be a real Hilbert space whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$ respectively. Let $K: H \to C(H)$, where C(H) denotes a family of all nonempty closed convex cones in H, be a multivalued mapping. Given a nonlinear mapping $T: H \to H$, we consider the quasi-complementarity problems of finding $u \in K(u)$ such that $T(u) \in K^*(u)$ and

$$\langle T(u), u \rangle = 0, \tag{1}$$

where $K^*(u)=\{w \in H: \langle w,z \rangle \geq 0 \text{ for all } z \in K(u)\}$, is a polar cone of $K(u)$.

We remark that the quasi-complementarity problem (1) and its generalizations so far have been considered and studied, by using variational inequality technique, only for a particular form of underlying set $K(u)$ which is $K(u)=K+m(u)$, where K is a closed convex cone in H and m is a point to point mapping from H into itself, [1, 3, 8, 9, 10, 11, 12, 13, 14].

The main purpose of this paper is to develop the existence and uniqueness theory of quasi-complementarity problem (1), where the underlying closed convex cone $K(u)$ is general, by making use of quasivariational inequality and fixed point techniques and to suggest an iterative algorithm for problem (1). We organize the rest of this paper as follows. In Section 2, we give some preliminaries that will be used throughout this paper. In Section 3, we show that quasi-complementarity problem (1) is equivalent to a fixed point problem. Further, we prove an existence and uniqueness result for quasi-complementarity problem (1). In Section 4, we suggest a unified and general iterative algorithm for obtaining the approximate solution of quasi-complementarity problem (1) and show that this approximate solution strongly converges to the exact solution.

2. PRELIMINARIES

We need the following definitions and lemmas:

Definition 2.1. A mapping $T: H \to H$ is called:

(i) α-strongly monotone, if there exists a constant $\alpha>0$ such that

$$\langle T(u)-T(v), u-v \rangle \geq \alpha \|u-v\|^2 \text{ for all } u,v \in H;$$

(ii) β-Lipschitz continuous, if there exists a constant $\beta>0$ such that

$$\|T(u)-T(v)\| \leq \beta \|u-v\| \text{ for all } u,v \in H;$$

(iii) contraction, if (ii) holds with $0<\beta<1$.

Definition 2.2. A multivalued mapping K: $H \to C(H)$ is said to be γ-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$\rho(K(u),K(v)) \leq \gamma \|u-v\| \text{ for all } u,v \in H,$$

where $\rho(K(u),K(v)) = \sup\{\|u-v\| : u \in K(u) \text{ and } v \in K(v)\}$

Lemma 2.1. ([6], Theorem 2.3). Let K be a nonempty closed convex subset of H. Then, given $z \in H$, we have $x = P_K(z)$ if and only if

$$\langle x-z, y-x \rangle \geq 0 \text{ for all } y \in K,$$

where P_K is a projection mapping from H onto K.

Lemma 2.2. ([6], Corolary 2.4). P_K: $H \to K$ is nonexpansive, i.e.

$$\|P_K(x)-P_K(y)\| \leq \|x-y\| \text{ for all } x,y \in H.$$

3. EXISTENCE RESULT

We need the following two technical lemmas in order to prove the main result of this section and the result of next section.

Lemma 3.1. Let K: $H \to C(H)$ be a multivalued mapping. Then $u \in K(u)$ is a solution of quasi-complementarity problem (1) if and only if $u \in K(u)$ satisfy the quasivariational inequality

$$\langle T(u), v-u \rangle \geq 0 \text{ for all } v \in K(u) \tag{2}$$

Proof. Its proof is similar to the of Lemma 3.1 in [9].

Lemma 3.2. $u \in K(u)$ is a solution of quasivariational inequality problem (2) if and only if $u \in K(u)$ is a solution of the fixed point problem

$$u = F(u), \tag{3}$$

where the mapping F: $H \to H$ is defined by, for some constant $\eta > 0$

$$F(u) = P_{K(u)}[u - \eta T(u)]. \tag{4}$$

Proof. Let $u \in K(u)$ be a solution of quasivariational inequality problem (2), i.e. we have $u \in K(u)$ such that

$$\langle T(u), v-u \rangle \geq 0 \text{ for all } v \in K(u),$$

and for some constant $\eta>0$, we have

$$\langle u-(u-\eta T(u)), v-u\rangle \geq 0 \text{ for all } v\in K(u).$$

By Lemma 2.1, we have

$$\begin{aligned} u &= P_{K(u)}[u-\eta T(u)] \\ &= F(u), \end{aligned}$$

i.e. $u\in K(u)$ is a solution of fixed point Problem (3). The converse part of this lemma can be easily proved by using the same arguments used above in reverse direction.

Now we prove the main result of this section.

Theorem 3.1. Let the multivalued mapping K: $H\to C(H)$ be a γ-Lipschitz continuous and let the mapping T: $H\to H$ be a α-strongly monotone and β-Lipschitz continuous. If

$$0 < \eta < \frac{2\alpha}{\beta^2}, \quad \gamma < 1-\sqrt{(1-2\eta\alpha+\eta^2\beta^2)}, \quad \alpha < \beta, \tag{5}$$

then $u\in K(u)$ is a unique solution of quasi-complementarity problem (1).

Proof. In order to prove this theorem it is enough to show that the mapping F: $H\to H$ defined by (4) has a unique fixed point. Now we consider

$$\begin{aligned} \|F(u)-F(v)\| &= \|P_{K(u)}[u-\eta T(u)]-P_{K(v)}[v-\eta T(v)]\| \\ &\leq \|P_{K(u)}[u-\eta T(u)]-P_{K(v)}[u-\eta T(u)]\| \\ &+ \|P_{K(v)}[u-\eta T(u)]-P_{K(v)}[v-\eta T(v)]\| \\ &\leq \rho(K(u),K(v))+\|P_{K(v)}[u-\eta T(u)]-P_{K(v)}[v-\eta T(v)]\| . \end{aligned}$$

By using Lemma 2.2 and γ-Lipschitz continuity of K, we have

$$\|F(u)-F(v)\| \leq \gamma\|u-v\| + \|u-v-\eta(T(u)-T(v))\| . \tag{6}$$

Since T is α-strongly monotone and β-Lipschitz continuous, it can be obtained that

$$\|u-v-\eta(T(u)-T(v))\|^2 \leq (1-2\eta\alpha+\eta^2\beta^2)\|u-v\|^2 . \tag{7}$$

Combining (6) and (7), we have

$$\|F(u)-F(v)\| \leq \{\gamma+\sqrt{(1-2\eta\alpha+\eta^2\beta^2)}\}\ \|u-v\|$$
$$\leq \theta\,\|u-v\| \ . \qquad (8)$$

where $\theta=\gamma+\sqrt{(1-2\eta\alpha+\eta^2\beta^2)}$. Since $\theta<1$ by condition (5), (8) implies that the mapping F is a contraction mapping and by Banach contraction principle, it has a unique fixed point. Hence by Lemma 3.1 and Lemma 3.2, we have that $u\in K(u)$ is a unique solution of quasi-complementarity problem (1).

4. CONVERGENCE ANALYSIS

In this section, on the basis of Lemma 3.1 and Lemma 3.2, we suggest a unified and general iterative algorithm for the quasi-complementarity problem (1) and show that the approximate solution obtained by iterative algorithm for quasi-complementarity problem (1) strongly converges to the exact solution.

Iterative Algorithm 4.1. For any given $u\in K(u)$, compute

$$u_{n+1} = (1-\lambda_n)u_n+\lambda_n P_{K(u_n)}[u_n-\eta T(u_n)] \ , \ n=1,2,3... \qquad (9)$$

for some constant $\eta>0$ and $0 \leq \lambda_n \leq 1$.

Theorem 4.1. Let K: $H\rightarrow C(H)$ and T: $H\rightarrow H$ be the same as in Theorem 3.1. let the condition (5) of Theorem 3.1 be hold and let $\sum_{n=1}^{\infty}\lambda_n$ be divergent. If u_{n+1} and u are the solution of Iterative algorithm 4.1 and quasi-complementarity problem (1), respectively, then u_{n+1} strongly converges to u in H.

Proof. Since u is a solution of quasi-complementarity problem (1) then by Lemma 3.1 and Lemma 3.2, we have that $u\in K(u)$ such that

$$u = P_{K(u)}[u-\eta T(u)].$$

Now we consider

$$\|u_{n+1}-u\| = \|(1-\lambda_n)(u_n-u)+\lambda_n\{P_{K(u_n)}[u_n-\eta T(u_n)]-P_{K(u)}[u-\eta T(u)]\}\|$$
$$\leq (1-\lambda_n)\|u_n-u\|+\lambda_n\{\rho(K(u_n),K(u))+\|u_n-u-\eta(T(u_n)-T(u))\|\} \ . \qquad (10)$$

Since K is a γ-Lipschitz continuous and T is a α-strongly monotone and β-Lipschitz continuous, then (10) becomes

$$\|u_{n+1}-u\| \le [(1-\lambda_n)+\lambda_n\{\gamma+\sqrt{(1-2\eta\alpha+\eta^2\beta^2)}\}]\|u_n-u\|$$
$$\le [(1-\lambda_n)+\lambda_n\theta]\|u_n-u\|,$$

where $\theta=\gamma+\sqrt{(1-2\eta\alpha+\eta^2\beta^2)}$. Since $\theta<1$ by condition (5)., we have

$$\|u_{n+1}-u\| \le [1-(1-\theta)\lambda_n]\|u_n-u\|.$$

By iteration, we get

$$\|u_{n+1}-u\| \le \prod_{i=1}^{n} [1-(1-\theta)\lambda_i]\ \|u_1-u\| . \tag{11}$$

Since $\sum_{n=1}^{\infty} \lambda_n$ diverges and $1-\theta>0$, we have $\prod_{i=1}^{n} [1-(1-\theta)\lambda_i] = 0$.

Thus (11) implies that u_{n+1} strongly converges to u.

REFERENCES

[1] CHANG, S.S., HUANG, N.J., Generalized Strongly Quasi-Complementarity Problems in Hilbert Spaces, Journal of Mathematical Analysis and Applications, Vol. 158 (1991), 194-202.

[2] COTTLE, R.W., DANTZIG, G.B., Complementarity Pivot Theory of Mathematical Programming, Linear Algebra Appl, Vol. 1 (1968), 103-125.

[3] DING, X.P., Generalized Strongly Nonlinear Quasivariational Inequalities, Journal of Mathematical Analysis and Applications, Vol. 173 (1993), 577-587.

[4] ISAC, G., Complementarity Problems, Lecture Notes in Mathematics, Vol. 1528, Springer-Verlag, 1992.

[5] KARAMARDIAN, D., Generalized Complementarity Problems, Journal of Optimization Theory and Applications, Vol. 8 (1971), 161-168.

[6] KINDERLEHRER, D., STAMPACCHIA, G., An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[7] LEMKE, C.E., Bimetrix Equilibrium Profits and Mathematical Programming, Management Sciences, Vol. 11 (1965), pp. 681-689.

[8] NOOR, M.A., Linear Quasi Complementarity Problems, Utilitas Mathematica, Vol. 27 (1985), 249-260.

[9] NOOR, M.A., Generalized Quasi-Complementarity Problems, Journal of Mathemalitcal Analysis and Applications, Vol. 120 (1986), 321-327.

[10] NOOR, M.A., The Quasi Complementarity Problem, Journal of Mathematical Analysis and Applications, Vol. 130 (1988), 344-353.

[11] NOOR, M.A., Nonlinear Quasi Complementarity Problems, Applied Mathematics Letters, Vol. 2 (1989), 251-254.

[12] NOOR, M.A., General Quasi Complementarity Problems, Mathematica Japonica, Vol. 36 (1991), 113-119.

[13] PANG, J.S., The Implicit Complementarity Problems, in Nonlinear Programming, 4 (1981) (Eds. Mangasarian, Meyer and Robinson) Academic Press, London, New York, 487-518.

[14] PANG, J.S., On the Convergence of a Basic Iterative Method for the Implict Complementarity Problem, Journal of Optimization Theory and Applications, Vol. 37 (1982), 149-162.