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GEOMETRICAL APPROACH TO BALANCED INCOMPLETE BLOCK DESIGNS

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ABSTRACT

Construction and analysis of balanced incomplete block design needs special care. There are several methods to construct balanced incomplete block designs. We tried to give relations between parameters of balanced incomplete block design and properties of projective geometry and related geometries. We offered some geometrical examples. Some of them can be considered original.

I. INTRODUCTION

There are several methods to construct BIB designs; some of them are Method of Symmetrically Repeated Difference, Method of Mutually Orthogonal Latin Squares, Method of Near-Rings, Hadamard Matrices Method [3].

$GF(p^n)$ has been used to obtain all BIB designs. Using finite analytical geometries, the construction can be done easily. For example, we can obtain at most p^n-1 orthogonal Latin Squares in p elements. Since $EG(2, p^n)$ is a kind of Affine planar, one can obtain $PG(2, p^n)$ by agumentation of Affine planar. $EG(2, p^n)$ and $PG(2, p^n)$ will represent several BIB designs.

Definition 1.1: Consider two sets T and B with their elements beings treatments and blocks, respectively. There are t treatments and b blocks in BIB design that satisfy the following conditions:

1- Each block has exactly k members,

2- Each treatment occurs in exactly r blocks,

3- Every pair of treatments occur in exactly λ blocks.

The parameters of BIB design satisfy the following equations
1) $n = bk = rt$,

2) $\lambda = r(k-1) / (t-1)$.

Where n is the total number of observations.

Definition 1.2: BIB design is said to be symmetrical if $t = b$.

II. FINITE PROJECTIVE GEOMETRY AND FINITE EUCLID GEOMETRY

There is $GF(p^n)$ of order p^n to every prime number p and every positive integer n. A point in the m dimensional finite projective geometry $PG(m, p^n)$ is an ordered set of $(m + 1)$ elements of $GF(p^n)$, not all of which are equal to 0.

Let $(x_1, x_2, \ldots, x_{m+1})$ and $(x'_1, x'_2, \ldots, x'_{m+1})$ are two sets. If

$$x_i = \gamma x'_i,\ i = 1, 2, \ldots, m + 1,\ \gamma \neq 0,\ \gamma \subset GF(p^n) \tag{2.1}$$

two sets represents the same point. Let $P_1 = (x_1, x_2, \ldots, x_{m+1})$ and $P_2 = (x'_1, x'_2, \ldots, x'_{m+1})$ be two distinct points. We define the line:

$$\gamma_1 P_1 + \gamma_2 P_2 = \gamma_1 x_1 + \gamma_2 x'_1 + \gamma_1 x_2 + \gamma_2 x'_2 + \cdots + \gamma_1 x_{m+1} + \gamma_2 x'_{m+1} \tag{2.2}$$

where $\gamma_1, \gamma_2 \subset GF(p^n)$ and at least one of the $\gamma_i \neq 0$.

The system of points and lines which are defined above is called analytic projective geometry of $GF(p^n)$ of m dimensions.

Number of different points in $PG(m, p^n)$ are given by

$$\frac{p^{n(m+1)}-1}{p^n-1} = 1 + p^n + \ldots + p^{mn}. \tag{2.3}$$

Each subset of $PG(m, p^n)$ of u dimensions is again $PG(u, p^n)$ and includes $1 + p^n + \ldots + p^{un}$ points. Each $PG(u, p^n)$ is a set of $(u + 1)$ linearly independent points. P_3 can be constituted by $(1 + p^n + \ldots + p^{mn})$ different selections. Similarly P_2 can be constituted by $p^n + \ldots + p^{mn}$ and P_3 can be constituted by $p^{2n} + \ldots + p^{mn}$ different selections. After choosing t points, we can take $(t + 1)$ points outside $PG(t-1, p^n)$, provided $t < u + 1$. We can find the number of ordered sets of $(u + 1)$ independent points in $PG(m, p^n)$. This number is

$$(1 + p^n + \ldots + p^{mn}) \ldots (p^{un} + \ldots + p^{mn}).$$

Similarly number of ordered sets of $(u + 1)$ independent points in $PG(u, p^n)$ is

$$(1 + p^n + \dots + p^{un}) \dots (p^{(u-1)n} + p^{un})p^{un}.$$

So number of PG(u, p^n) in PG(m, p^n) is

$$\frac{(1 + p^n + \dots + p^{mn}) \dots (p^{un} + \dots + p^{mn})}{(1 + p^n + \dots + p^{un}) \dots (p^{(u-1)n} + p^{un})\, p^{un}} . \qquad (2.4)$$

Number of PG(s, p^n) in PG(m, p^n) that include a given PG(u, p^n) is

$$\frac{(p^{(u+1)n} + \dots + p^{mn}) \dots (p^{sn} + \dots + p^{mn})}{(p^{(u+1)n} + \dots + p^{sn}) \dots + (p^{(s-1)n} + p^{sn})\, p^{sn}} . \qquad (2.5)$$

We can regard blocks and treatments of BIB design as the lines and the points of PG(m, p^n).

Theorem 1. The subset PG(s, p^n) of PG(m, p^n) constitutes a BIB design with the following parameters [2]:

$$b\,(s, m, p^n) = \frac{(1 + p^n + \dots + p^{mn}) \dots (p^{sn} + \dots + p^{mn})}{(1 + \dots + p^{sn}) \dots (p^{(s-1)n} + p^{sn})\, p^{sn}} ,$$

$$t\,(m, p^n) = 1 + p^n + \dots + p^{mn},$$

$$k\,(s, p^n) = 1 + p^n + \dots + p^{sn},$$

$$r\,(s, m, p^n) = \frac{(p^n + \dots + p^{mn}) \dots (p^{sn} + \dots + p^{mn})}{(p^n + \dots + p^{sn}) \dots (p^{(s-1)n} + p^{sn})\, p^{sn}} , \qquad (2.6)$$

$$\lambda\,(s, m, p^n) = \begin{cases} 1 & \text{for } s = 1 \\ \dfrac{(p^{2n} + \dots + p^{mn}) \dots (p^{sn} + \dots + p^{mn})}{(p^{2n} + \dots + p^{mn}) \dots (p^{(s-1)n} + p^{sn})p^{sn}} & \text{for } s > 1. \end{cases}$$

If we delete any PG(m–1, p^n) from a PG(m, p^n), we can obtain EG(m, p^n).

Theorem 2: The subspace EG(s, p^n) of EG(m, p^n) constitute a BIB design with parameters [2]:

$$b = b\,(s, m, p^n) - b\,(s, m-1, p^n)$$

$$t = p^{mn},$$

$$k = p^{sn},$$

$$r = r\,(s, m, p^n),$$

$$\lambda = \lambda\,(s, m, p^n).$$

Theorem 3. (Bruck-Rayser Theorem): There does not exist a projective plane of order p^n if $p^n = 1 \pmod 4$ or $p^n = 2 \pmod 4$ and p^n is not a sum of two nonnegative integers [1].

According to the Bruck–Ryser theorem, projective planes of some order do not exist.

EXAMPLES

1. For $b = t = 4$, $r = k = 3$, $\lambda = 2$, BIB design can be shown as in Table 1.

Table 1. BIB design for $b = t = 4$, $r = k = 3$, $\lambda = 2$

Treatments	BLOCKS			
	1	2	3	4
A	X		X	X
B		X	X	X
C	X	X	X	
D	X	X		X

According to the Bruck–Ryser theorem, the BIB design above can not be a projective plane but can be shown as in Figure 1.

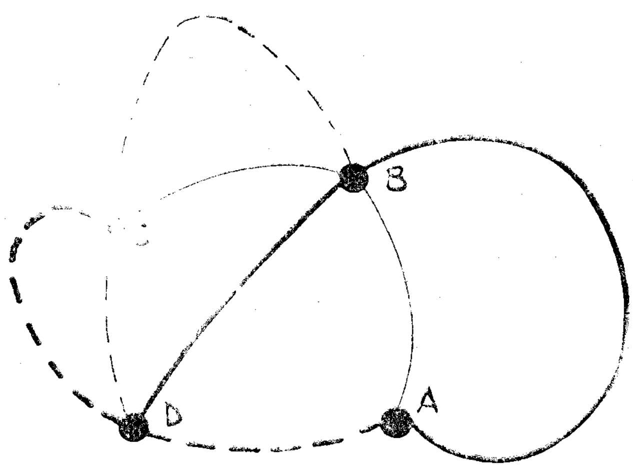

Figure 1

2. For $t = b = 13$, $k = r = 4$, $\lambda = 1$ BIB design can be shown as in Table 2.

Table 2. BIB design for t = b = 13, k = r = 4, λ = 1

Treatments	BLOCKS												
	1	2	3	4	5	6	7	8	9	10	11	12	13
1	X	X	X	X									
2	X				X	X						X	
3		X			X		X	X					
4	X						X		X	X			
5		X				X			X		X		
6			X		X					X	X		
7			X			X	X						X
8	X							X			X		X
9		X								X		X	X
10			X					X	X			X	
11				X		X		X		X			
12				X			X				X	X	
13				X	X				X				X

Using Theorem 1, one can obtain the BIB design as in Figure 2, which can be represented as PG (2, 3). The points and the lines of PG (2, 3) are:

Points

P_1: 100	P_5: 101	P_9: 10–1	P_{13}: 0–11
P_2: 010	P_6: 011	P_{10}: –111	
P_3: 001	P_7: 111	P_{11}: 1–11	
P_4: 110	P_8: 1–10	P_{12}: 11–1	

Lines	The points on the lines			
[001] :	P_1	P_2	P_4	P_8
[010] :	P_1	P_3	P_5	P_9
[0–11] :	P_1	P_6	P_7	P_{10}
[011] :	P_1	P_{11}	P_{12}	P_{13}
[100] :	P_2	P_3	P_6	P_{13}

[10–1] :	P_2	P_5	P_7	P_{11}
[101] :	P_2	P_9	P_{10}	P_{12}
[1–10] :	P_3	P_4	P_7	P_{12}
[110]	P_3	P_8	P_{10}	P_{11}
[–111] :	P_4	P_5	P_{10}	P_{13}
[1–11] :	P_4	P_6	P_9	P_{11}
[11–1]	P_5	P_6	P_8	P_{12}
[111] :	P_7	P_8	P_9	P_{13}

The BIB design is a projective plane which is shown in Figure 2.

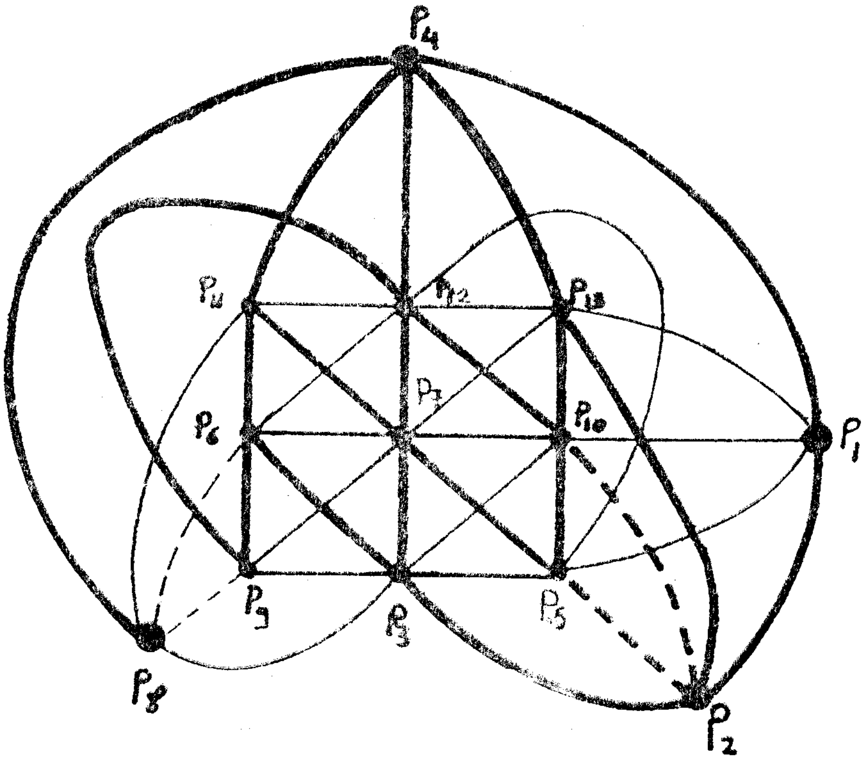

Figure 2

Let's delete the line [001] and all its points from PG (2, 3). Using Theorem 2, one can obtain the BIB design as in Table 3 which can be represented as EG (2, 3).

3. For $t = 9$, $b = 12$, $r = k = 4$, $\lambda = 1$, BIB design can be shown as in Table 3.

Tablo 3. BIB design for $t = 9$, $b = 12$, $r = k = 4$, $\lambda = 1$

Treatments	BLOCKS											
	2	3	4	5	6	7	8	9	10	11	12	13
3	X			X			X	X				
5	X				X				X		X	
6		X		X						X	X	
7		X			X		X					X
9	X					X				X		X
10		X				X		X	X			
11			X		X			X		X		
12			X			X	X				X	
13			X	X					X			X

The BIB design which is related to Table 3 is a EG (2,3) which is shown in Figure 3.

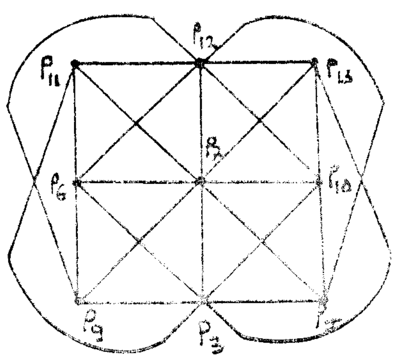

Figure 3

The designs in above examples are some of the well known designs in statistics and employed in experimental work.

ÖZET

Bu çalışmada projektif ve Euclid geometrinin özelliklerinden faydalanarak dengeli tamamlanmamış blok deney düzeninin geometrik gösterimi verilmiştir. Projektif geometrinin doğruları ve noktaları sırasıyla dengeli tamamlanmamış blok deney düzeninin blokları ve işlemlerine karşılık gelmektedir.

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