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PARALLEL PROJECTION AREA AND HOLDITCH'S THEOREM

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ABSTRACT

In this paper, the parallel projection area of a closed spatial curve formed under the motion $B(c_1)$ defined along the closed spherical curve c_1 [12] have been calculated. After that Holditch's Theorem [7] and its some corollaries which is well-known [3] have been generalized to closed spatial curve.

1. INTRODUCTION

The study of one-parameter closed motions become an interesting subject in kinematics after the work of Jacob Steiner [11] and H. Holditch [7].

During the second half of the nineteenth century, there appeared many publications about Steiner's and Holditch's Theorems; for example: C. Leudesdorf [9, 10] and A.B. Kempe [8].

After the work of Steiner and Holditch the first study about spherical motions was given by E.B. Elliott [2, 3, 4]. Another study in this field was also given by H.R. Müller [11]. H.H. Hacısalihoğlu [6] obtained a formula which is equivalent to the Holditch formula. R. Güneş and S. Keleş [5], using the area formula and the area vector given by W. Blaschke [1] and H.R. Müller [11], respectively, obtained the formula given by H.H. Hacısalihoğlu by a different method.

H. Pottmann [12] defined the spherical motion along a curve on a sphere and also he gave the parallel projection area of the spherical indicator using the parallel projection area vector.

In this study, H. Holditch's Theorem, which is well-known for one-parameter closed planar motions, was generalized to the closed spatial motions by using the area vector which was described by H.R. Müller

[11] and parallel projection area formula which was given by H. Pottmann [12] and some results were obtained.

2. SPHERICAL CURVES

Let a c_1-curve of class C^2 on a unit sphere K^1 of 3-dimensional Euclidean space is given by

$$\vec{e}_1 : t \in I \subset IR \rightarrow \vec{e}_1(t) \in IR^3,\ \|\vec{e}_1\| = 1\ ,\ \vec{e}_1 \in C^2(I)$$
$$c_1 = \vec{e}_1(I) \tag{1}$$

where C^2 denotes the set of twice continuously differentiable curves.

Let us consider the sphere K coinciding with K^1 to be $K = K^1$, where K^1 is a fixed sphere and K is a moving sphere with respect to K^1. In this case the curve c_1 on K^1 defines an accompanying motion $B(c_1)$. The end point $E_1(t) \in K$ of the vector $\vec{e}_1(t)$ lies on the curve c_1 which is always tangent any constant big circles at $E_1(t)$ as illustrated in Figure 1.

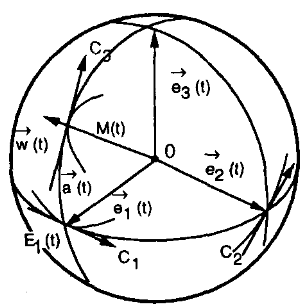

Fig. 1

At the initial time, given an orthonormal frame $\{0;\vec{e}_1(t_0),\vec{e}_2(t_0),\vec{e}_3(t_0)\}$, which are rightly linked to the origin point of the moving sphere K, is defined as

$$\vec{e}_2(t_0) = \frac{\dot{\vec{e}}_1(t_0)}{\|\dot{\vec{e}}_1(t_0)\|}\ ,\ \vec{e}_3(t_0) = \vec{e}_1(t_0) \wedge \vec{e}_2(t_0) \tag{2}$$

During the closed motion $B(c_1)$, a frame $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ can be defined for the point $E_1(t)$ of the curve c_1 at $t=t_1$ with the help of the limit $t\to t_1$. The closed motion $B(c_1)$ defined along the curve c_1 having completely the inflection point 2n ($n\in \mathbb{N}\cup\{0\}$) of c_1 is known as a closed motion [12].

Derivative equations of the moving frame can be written in the matrix form as

$$\begin{bmatrix} \dot{\vec{e}}_1 \\ \dot{\vec{e}}_2 \\ \dot{\vec{e}}_3 \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & \mu \\ 0 & -\mu & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} \tag{3}$$

In equation (3), Darboux rotation vector is

$$\vec{w} = \mu\vec{e}_1 + \lambda\vec{e}_3 \tag{4}$$

Hence equation (3) can be also written in the following vector form

$$\dot{\vec{e}}_i = \vec{w} \wedge \vec{e}_i \ , \ (i=1,2,3) \tag{5}$$

If $\vec{w} \neq 0$ then we write the vectors

$$\vec{p}_1 = \frac{\vec{w}}{\|\vec{w}\|} \ , \ \vec{p}_2 = -\frac{\vec{w}}{\|\vec{w}\|} \ .$$

The vectors $\vec{p}_1$ and $\vec{p}_2$ define two constant Pole points M_i, $i=1,2$, which are symmetric with respect to the origin point O, of the closed motion $B(c_1)$ at the time t. The points M_1 and M_2 are on the big circle q in the plane $(\vec{e}_1,\vec{e}_3)$.

In the case of $\vec{w} = \vec{0}$, an instantaneous standstill of the closed motion $B(c_1)$ is happened. During the closed motion $B(c_1)$, q is rolled without sliding on spherical evaluate q^1 of the curve c_1.

Now, we discuss the close curve in $\mathbb{R}^3$ by the closed motion $B(c_1)$. Because, there is a very close relationship between spherical accompanying curves and curves theory in $\mathbb{R}^3$.

Let us now consider the closed accompanying motion $B(c_1)$ that leaves origin point of a moving space R with respect to a fixed space R^1 unchanged. In such a way that we again have the derivative equations given by (3) for a space curve k in $\mathbb{R}^3$.

The end points of vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ draw, respectively, spherical tangent indicator c_1, principal normal indicator c_2 and binormal indicator c_3 of the space curve k. A closed orientated periodic curve $k \subset IR^3$ of the class C^1 is given as

$$\begin{gathered} \vec{x}: t \in I = [0;L) \subset IR \rightarrow \vec{x}(t) \in IR^3,\ \vec{x} \in C^1(R) \\ \vec{x}(t+L) = \vec{x}(t) \\ k = \vec{x}(I) \end{gathered} \tag{7}$$

3. PARALLEL PROJECTION AREA AND HOLDITCH'S THEOREM

Definition 3.1. Let c(X) be a closed curve in 3-dimensional Euclidean space and X be a point on c(X). The vector satisfying

$$\vec{V}_x = \oint \vec{x}(t) \wedge \dot{\vec{x}}(t)dt \tag{8}$$

is called the area vector of the curve c(X) [11] in which $\vec{x}$ is the position vector of X.

Theorem 3.2. Let c(X) be a closed curve in 3-dimensional Euclidean space and X be a point on c(X). The projection area [11] of the planar region occurred by taking orthogonal projection onto a plane in the direction the unit vector $\vec{n}$ of c(X) is

$$2F_{X^n} = \langle \vec{n}, \vec{V}_x \rangle\ . \tag{9}$$

Theorem 3.3. (Holditch's Theorem [7]). If a chord of a closed curve, of constant length a+b, be divided into two parts of lengths a, b, respectively the difference between the areas of the closed curve, and of the locus of the dividing point, will be

$$F = F_\Gamma - F_X = \pi ab$$

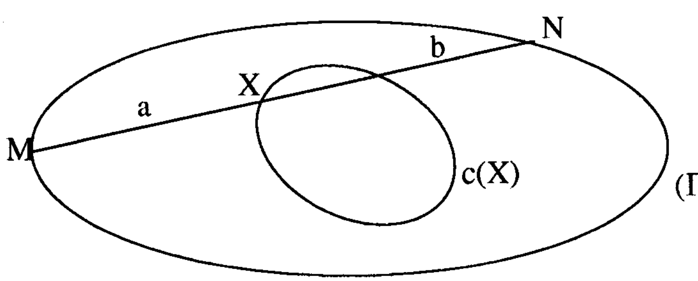

Fig. 2

After the above preparations, we can calculate the area vector $\vec{V}_x$ of the orbit c(X), on IR^1, of the fixed point $X \in R$, during the closed accompanying motion $B(c_1)$, along a closed spherical curve c_1. Let I = [0,L) be the period interval of an instantaneous closed accompanying motion $B(c_1)$.

The position vector $\vec{x}$ of a point $X \in R$ with respect to the vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ of K can be written as

$$\vec{x}(t) = x_1\vec{e}_1(t) + x_2\vec{e}_2(t) + x_3\vec{e}_3(t) \tag{10}$$

where x_1, x_2 and x_3 are constant coordinates of X. From (8), the area vector of the orbit c(X) drawn by a point $X \in R$ under the closed accompanying motion $B(c_1)$ is obtained as

$$\vec{V}_x = \sum_{i=1}^{3} x_i^2\vec{V}_{E_i} + 2\sum_{\substack{i,k=1\\ i<k}}^{3} x_i x_k \vec{V}_{E_{ik}} \tag{11}$$

where

$$\vec{V}_{E_i} = \int_0^L \vec{e}_i(t) \wedge \dot{\vec{e}}_i(t)dt \quad , \quad \vec{V}_{E_{ik}} = \frac{1}{2}\int_0^L (\vec{e}_i(t) \wedge \dot{\vec{e}}_k(t) + \vec{e}_k(t) \wedge \dot{\vec{e}}_i(t))dt \tag{12}$$

We have

$$\int_0^L \dot{\vec{e}}_i(t)dt = 0 \tag{13}$$

since $B(c_1)$ is a closed accompanying motion. By using equation (3), (12) and (13) we obtain

$$\vec{V}_{E_1} = \int_0^L \lambda\vec{e}_3(t)dt \ , \ \vec{V}_{E_3} = \int_0^L \mu\vec{e}_1(t)dt \ , \ \vec{V}_{E_2} = \vec{V}_{E_1} + \vec{V}_{E_3}$$

and

$$\vec{V}_{E_{12}} = 0 \ , \ \vec{V}_{E_{23}} = 0 \ , \ \vec{V}_{E_{13}} = -\int_0^L \lambda\vec{e}_1(t)dt = -\int_0^L \mu\vec{e}_3(t)dt \tag{14}$$

Substitution of (14) into (11) gives

$$\vec{V}_X = \sum_{i=1}^{3} x_i^2\vec{V}_{E_i} + 2x_1x_3\vec{V}_{E_{13}} \tag{15}$$

where $\vec{V}_{E_{13}}$ is the mixed area vector of the curve c_1 and the spherical curve c_3 being a spherical distance $\frac{\pi}{2}$ to c_1.

The mixed area of orthogonal projections of curves c_1 and c_3 in the direction $\vec{n}$ ($\|\vec{n}\| = 1$) is

$$2F_{E_{13}}^{n} = \langle \vec{n}, \vec{V}_{E_{13}} \rangle \tag{16}$$

Corollary 3.4. Let F_{X^n} be the projection area of the planar region occurred by taking orthogonal projection onto a plane of c(X) and F_{X^p} be the projection area of the planar region happened by projecting onto same plane of c(X) in any direction. From here

$$F_{X^n} = \cos\theta \, F_{X^p} \tag{17}$$

where θ is the angle between two image planes.

Theorem 3.5. Let the closed accompanying motion $B(c_1)$, along the curve c_1 on a unit sphere of the class C^2, be given. Then, the orientated area F_{X^p} of parallel projection of orbit c(X) of a constant point $X \in R$, in terms of the orientated areas $F_{E_{13}^p}$ and $F_{E_i^p}$, i=1,2,3 can be obtained as

$$F_{X^p} = 2x_1x_3F_{E_{13}^p} + \sum_{i=1}^{3} x_i^2 F_{E_i^p} \tag{18}$$

Theorem 3.6. The orientated projection area F_{X^p} of the planar region occurred by parallel projection of the curve c(X) drawn by a constant point $X \in R$ during the closed accompanying the motion $B(c_1)$ is a quadratic form according to coordinates x_i, i=1,2,3.

If the coordinate systems are chosen properly that is, if a proper rotation is applied, from eq. (18) the orientated projection area F_{X^p} is obtained as

$$F_{X^p} = \sum_{i=1}^{3} x_i^2 F_{E_i^p} \tag{19}$$

Let X and Y be two different fixed points in the moving space R. Suppose that Z is a point with the components

$$z_i = \lambda x_i + \mu y_i \quad , \quad \lambda + \mu = 1 \ , \ 1 \le i \le 3 \ , \tag{20}$$

on the straight line XY. The point Z has an orbit c(Z) in R^I during the closed accompanying motion $B(c_1)$. The area bounded by the orthogonal projection of the closed curve c(Z) on the plane is

$$F_{Z^P} = \lambda^2 \sum_{i=1}^{3} F_{E_i^P} x_i^2 + 2\lambda\mu \sum_{i=1}^{3} F_{E_i^P} x_i y_i + \mu^2 \sum_{i=1}^{3} F_{E_i^P} y_i^2 \tag{21}$$

From Eq.(12), mixed area vector of curves c(X) and c(Y) can be obtained as

$$\vec{V}_{XY} = \sum_{i=1}^{3} x_i y_i \vec{V}_{E_i} + (x_1 y_3 + y_1 x_3)\vec{V}_{E_{13}} \tag{22}$$

The orientated mixed area $F_{X^P Y^P}$ of the planar region occurred by taking parallel projection of curves c(X) and c(Y) onto a plane is

$$F_{X^P Y^P} = \sum_{i=1}^{3} x_i y_i F_{E_i^P} + (x_1 y_3 + y_1 x_3) F_{E_{13}^P} \tag{23}$$

By using Eq.(23) in Eq.(21) we obtain

$$F_{Z^P} = \lambda^2 \sum_{i=1}^{3} x_i^2 F_{E_i^P} + 2\lambda\mu \left\{ F_{X^P Y^P} - (x_1 y_3 + y_1 x_3) F_{E_{13}^P} \right\} + \mu^2 \sum_{i=1}^{3} x_i^2 F_{E_i^P} \tag{24}$$

Since

$$\sum_{i=1}^{3} F_{E_i^P}(x_i - y_i)^2 = F_{X^P} - 2\left\{ F_{X^P Y^P} - (x_1 y_3 + y_1 x_3) F_{E_{13}^P} \right\} + F_{Y^P}$$

and

$$\lambda + \mu = 1 \quad , \quad \lambda^2 = 1 - \lambda\mu \quad , \quad \mu^2 = \mu - \lambda\mu \tag{25}$$

from Eq.(24), with some manipulations, we can see that

$$F_{Z^P} = \lambda F_{X^P} + \mu F_{Y^P} - \lambda\mu \sum_{i=1}^{3} F_{E_i^P}(x_i - y_i)^2 \tag{26}$$

The distance between the points X and Y can be given by the metric

$$D^2(X,Y) = \varepsilon \sum_{i=1}^{3} F_{E_i^P}(x_i - y_i)^2 \tag{27}$$

such that $\varepsilon = \pm 1$.

For distinct points X, Y and Z lying on the same straight line we can write

$$D(X,Y) = D(X,Z) + D(Z,Y) \tag{28}$$

Eq.(28) can be re-written as

$$\frac{D(X,Z)}{D(X,Y)} + \frac{D(Z,Y)}{D(X,Y)} = 1$$

Since $\lambda+\mu = 1$, we can take λ and μ as

$$\lambda = \frac{D(X,Z)}{D(X,Y)} \quad , \quad \mu = \frac{D(Z,Y)}{D(X,Y)} \tag{29}$$

Eq.(26) can be written as

$$F_{Z^P} = \frac{1}{D(X,Y)}\ \varepsilon \left\{F_{X^P}D(X,Z) + F_{Y^P}D(Z,Y)\right\} - \varepsilon D(X,Z)D(Z,Y) \tag{30}$$

Suppose that the fixed point X and Y in the moving space draw same closed curve (Γ) during the closed accompanying motion $B(c_1)$. In this case, $\vec{V}_X = \vec{V}_Y$ and that's why $F_{X^P} = F_{Y^P}$. Thus the Eq.(30) reduces to

$$F_{X^P} - F_{Z^P} = \varepsilon D(X,Z)D(Z,Y) \tag{31}$$

which gives generalized Holditch's Theorem.

Let ℓ be a fixed straight line in the moving space R and let four arbitrary fixed points M, X, Y and N be on the line ℓ. During the closed accompanying motion $B(c_1)$, while the points M and N move on the same curve (Γ), the points X and Y draw the different curves c(X) and c(Y).

Corollary 3.7. Let F and F^1 be the areas between the parallel projections of the curves (Γ) and c(X) and of the curves (Γ) and c(Y), respectively. Then the ratio F/F^1 depends only on the relative positions of these four points.

Proof: According to (31), the area F^1 between the projection of the curves (Γ) and c(Y) is

$$F^1 = F_{M^P} - F_{Y^P} = \varepsilon D(M,Y)D(Y,N)$$

and the area F between the projection of the curves (Γ) and c(X) is

$$F = F_{M^P} - F_{X^P} = \varepsilon D(M,X)D(X,N)$$

Then, joining the last two equalities the ratio F/F^1 can be obtained as

$$\frac{F}{F^1} = \frac{D(M,X)D(X,N)}{D(M,Y)D(Y,N)} \text{ or } \frac{F}{F^1} = \left(\frac{D(M,X)}{D(M,Y)}\right)^2 \frac{D(M,Y)D(X,N)}{D(M,X)D(Y,N)} \tag{32}$$

The invariant (32) does not depend on the curve (Γ) and length of MN. It depends only on the choice of the points X and Y on MN. Since $X \neq Y$, it follows that $\frac{D(M,Y)}{D(M,X)} \neq 1$. Denote $\beta = \frac{D(M,Y)D(X,N)}{D(M,X)D(Y,N)}$. β is the cross ratio of the four points M, X, Y and N, i.e. $\beta = (MXYN)$.

Corollary 3.7 is the re-stated form of the corollary which has been given for one-parameter closed planar motions. Thus the corollary in [6] is generalized to the points of space and spatial motions.

Theorem 3.8. Let M, N, A and B be four different fixed points in the moving space R. Suppose that the line segments MN and AB meet at the point X. Then the pairs of the points M, N and A, B are on the same curve or the areas bounded by the parallel projection of the closed orbits of the pairs on the plane P are equal if and only if

$$D(M,X)D(X,N) = D(A,X)D(X,B).$$

Corollary 3.9. Let M, N, A and B be four different fixed points in the moving space R. Suppose that the line segments MN and AB meet at the point X. If the points M, N and B are on the same curve (Γ). Then it is $F_{M^P} = F_{A^P}$.

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