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ON THE SEQUENCE OF FOURIER COEFFICIENTS BY $\|T\|.C_1$ METHOD

RAJIV SINHA* and HEMANT KUMAR**

* *Department of Mathematics, S.M. College, Chandausi (U.P.), India.*
** *Department of Physics, S.M. College, Chandausi (U.P.), India.*

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ABSTRACT

Mohanty and Nanda [2] were the first to establish a result for (C,1) i.e. C_1-summability of the sequence $\{n\, B_n(x)\}$. Varshney [5] established a theorem on $(N, \frac{1}{n+1})C_1$ summability. In the present paper we have discussed $(a_{n,k})C_1$-summability of the sequence $\{n\, B_n(x)\}$ which includes the result due to Tripathi and Singh [7].

1. Let $\Sigma\, U_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\|T\| \equiv (a_{n,k})$ be infinite triangular matrix with real constants. Then sequence to sequence transformation.

$$t_n = \sum_{k=0}^{n} a_{n,k} S_k \qquad n = 0,1,2,...$$

defines the T-transform of the sequence $\{S_n\}$. Recall that the matrix elements $a_{n,k} = 0$ for each $K > n$, then the matrix is called triangular.

The series $\Sigma\, U_n$ is said to be T-summable to S, if $\lim_{n\to\infty} t_n = S$.

The regularity conditions for T-method are:

(1) There exists a constant K such that $\sum_{k=0}^{n} |a_{n,k}| < K$, for each n;

(2) For ever K, $\lim_{n\to\infty} a_{n,k} = 0$; and

(3) $\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1.$

The matrix T reduces to Nörlund matrix generated by the sequence of coefficients $\{p_n\}$ if

$$a_{n,k} = \begin{cases} p_{n-k}/P_n & , \text{ if } K \le n \\ 0 & , \text{ if } K > n \end{cases}$$

where $P_n = \sum_{r=0}^{n} p_r \neq 0$.

If the method of summability $\|T\|$ is applied to Cesàro means of order one, another method of summability $\|T\|.C_1$ is obtained.

2. Let f(x) be a periodic function with period 2π and integrable in the sense of Lebesgue over an interval $(-\pi,\pi)$. Let the Fourier series of f(x) be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x) \tag{2.1}$$

and then the conjugate series of (2.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) \tag{2.2}$$

We write

$$\psi(t) = f(x+t) - f(x-t) - L,$$

(where L is some constant),

$$\psi_1(t) = \int_0^t \psi(u)\, du \ ;$$

$$A_{n,\tau} = \sum_{k=\tau}^{n} a_{n,k} \ ;$$

and $\tau = [1/t]$ the integral part of $1/t$.

3. Mohanty and Nanda [2] proved the following theorem:

Theorem A: If

$$\psi(t) = O\left(\frac{1}{\log(1/t)}\right) \text{ as } t \to 0,$$

and $a_n = O(n^{-\delta})$; $b_n = O(n^{-\delta})$, $0 < \delta < 1$, then the sequence $\{n\, B_n(x)\}$ is (C,1) summable to the value L/π.

Varshney [5] generalized the above theorem of Mohanty and Nanda [2] which was later on extended by Tripathi and Singh [7] in the following form:

Theorem B: Let a function p(u), tending to infinity with u and a sequence $\{p_n\}$ be defined as follows in terms of p(u), monotonic decreasing and strictly positive for $u \geq 0$

$$P(u) \equiv \int_0^u p(x)\, dx \ , \ p_n \equiv p(n) \tag{3.1}$$

$$\psi_1(t) = O\left(\frac{t}{\varepsilon(1/t)}\right) \ , \text{ as } t \to +0 \tag{3.2}$$

$\varepsilon(t)$ being positive non-decreasing with t and

$$\int_1^n \frac{P(x)}{x\ \varepsilon(x)}\, dx = O(P_{(n)}) \ , \tag{3.3}$$

then the sequence $\{n\ B_n(x)\}$ is summable $(N,p_n).C_1$ to the value L/π. The object of this paper is to generalize the above theorem of Tripathi and Singh [8] for $\|T\|.C_1$ summability. However, our theorem is as follows:

Theorem: Let $\|T\| \equiv (a_{n,k})$ be an infinite triangular matrix with $a_{n,k} \geq 0$ and $a_{n,k}$ be defined by $a_{n,k} \equiv a_n(k)$, $a_n(u)$ being a strictly positive monotonic non-increasing function and

$$A(n,n-u) = \int_0^u a_n(n-t)\, dt \to 1 \text{ as } n \to \infty \text{ for fixed } u \geq 0. \tag{4.1}$$

Let $\varepsilon(t)$ be positive non-decreasing function of t.

If

$$\psi_1(t) = O\left(\frac{t}{\varepsilon(1/t)}\right) \ , \text{ as } t \to +0. \tag{4.2}$$

and

$$\int_1^n \frac{A(n,n-u)}{u\ \varepsilon(u)}\, du = O(1) \ , \text{ as } n \to \infty. \tag{4.3}$$

then the sequence $\{n\ B_n(x)\}$ is summable $\|T\|.C_1$ to the value L/π.

We note that (4.2) and (3.2) are same while conditions (4.1) and (4.3) in the case of $(N,p_n).C_1$ summability reduce to conditions (3.1) and (3.3), respectively.

5. For the proof of the theorem we require the following lemmas:

Lemma 1: (Kishore and Hotta [8]).

If $\{a_{n,k}\}_{k=0}^{n}$ is non-negative and non-decreasing with respect to k, then for $0 \le a < b \le \infty$, $0 \le t \le \pi$ and any n,

$$|\sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)t}| \le K\ A_{n,n-\tau} \tag{5.1}$$

Lemma 2: (Mittal [10]). If $0 \le t \le 1/n$, then

$$|Q_n(t)| \equiv \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \left(\frac{\sin\ kt}{kt^2} - \frac{\cos\ kt}{t} \right) = O(n) \tag{5.2}$$

Lemma 3: (Mittal [10]). If $0 < t \le \pi$, then

$$|Q_n(t)| = O\left(\frac{A_{n,n-\tau}}{t}\right) \tag{5.3}$$

6. Proof of the Theorem: If we denote the (C,1) transform of the sequence $\{n\ B_n(x)\}$ by t_n, we have after Mohanty and Nanda [2]

$$\begin{aligned} t_n - \frac{L}{\pi} &= \frac{1}{n} \sum_{r=1}^{n} r\ B_r(x) - L/\pi \\ &= \frac{1}{\pi} \int_0^{\pi} \psi(t) \left(\frac{1}{4n} \frac{1}{2} \frac{\sin\ nt}{\sin^2 \frac{1}{2} t} - \frac{\cos\ nt}{\tan \frac{1}{2} t} \right) dt \\ &+ \frac{1}{2\pi} \int_0^{\pi} \psi(t) \sin\ nt\ dt + O(1) \\ &= \frac{1}{\pi} \int_0^{\pi} \psi(t) \left[\frac{\sin\ nt}{nt^2} - \frac{\cos\ nt}{t} \right] dt + O(1). \end{aligned}$$

by Riemann-Lebesgue theorem.

On account of the regularity of the method of summability we have to show that under our assumptions

$$I = \int_0^{\pi} \frac{\psi(t)}{\pi} \sum_{k=0}^{n} a_{n,k} \left[\frac{\sin\ kt}{kt^2} - \frac{\cos\ kt}{t} \right] dt = O(1) \tag{6.1}$$

as $n \to \infty$.

From Lemma 2 we write

$$Q_n(t) \equiv \frac{1}{\pi} \sum_{k=1}^{n} a_{n,k} \left[\frac{\sin\ kt}{kt^2} - \frac{\cos\ kt}{t} \right]$$

Therefore

$$
\begin{aligned}
I &= \int_0^{\pi} \psi(t)\, Q_n(t)\, dt \\
&= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right) \psi(t)\, Q_n(t)\, dt \\
&= I_1 + I_2 + I_3 \text{ , say, where } 0 < \delta < \pi.
\end{aligned}
$$

Now by Lemma 2 and hypothesis (4.2), we have

$$
\begin{aligned}
I_1 &= \int_0^{1/n} \psi(t)\, Q_n(t)\, dt \\
&= O\left(n \int_0^{1/n} |\psi(t)|\, dt\right) \\
&= O\left(n\, O\left(\frac{1}{n\, \varepsilon(n)}\right)\right) \\
&= O\left(\frac{1}{\varepsilon(n)}\right) .
\end{aligned}
$$

Therefore $$I_1 = O(1) \text{ , as } n \to \infty. \tag{6.2}$$

Again by Lemma 3,

$$
\begin{aligned}
I_2 &= \int_{1/n}^{\delta} \psi(t)\, Q_n(t)\, dt \\
&= O\left(\int_{1/n}^{\delta} \left(|\psi(t)|\, \frac{A_{n,n-\tau}}{t}\right) dt\right) \\
&= O\left(\int_{1/n}^{\delta} |\psi(t)| \left(\frac{A_{n,n-\tau}}{t\, \varepsilon(t)}\, \varepsilon(t)\right) dt\right) \\
&= O\left(\psi_1(t)\, \frac{A_{n,n-\tau}}{t}\right)_{1/n}^{\delta} \\
&\quad + O \int_{1/n}^{\delta} \psi_1(t)\, \frac{d}{dt}\left(\frac{A_{n,n-\tau}}{t\, \varepsilon(1/t)}\, \varepsilon(1/t)\right) dt \\
&= O\left(\frac{1}{\varepsilon(n)}\right) + O\left(\frac{1}{\varepsilon(n)}\right) \\
&\quad + O(1) \int_{1/n}^{\delta} \left(\frac{t}{\varepsilon(1/t)}\, \frac{A_{n,n-\tau}}{t\, \varepsilon(1/t)}\right) \frac{d}{dt}\, (\varepsilon(1/t)) \\
&\quad + O(1) + O(1) \int_{1/n}^{\delta} 0 \left(\frac{t}{\varepsilon(1/t)}\right) \frac{d}{dt} \left(\frac{A_{n,n-\tau}}{t\, \varepsilon(1/t)}\right) .
\end{aligned}
$$

$$= O\left(\frac{t}{\varepsilon(n)}\right) + O\left(\frac{t}{\varepsilon(n)}\right)$$

$$+ O(1)\int_{1/n}^{\delta} \frac{\frac{d}{dt}\,\varepsilon(1/t)}{(\varepsilon(1/t))^2} + 0(1)\int_{1/n}^{\delta} t\,\frac{d}{dt}\cdot\left(\frac{A_{n,n-\tau}}{t\,\varepsilon(1/t)}\right)$$

$$= O\left(\frac{1}{\varepsilon(n)}\right) + O\left(\frac{1}{\varepsilon(n)}\right) + O(1)\left[\frac{1}{\varepsilon(1/t)}\right]_{1/n}^{\delta}$$

$$+ O(1)\left(\left[\frac{t\,A_{n,n-\tau}}{t\,\varepsilon(1/t)}\right]_{1/n}^{\delta} - \int_{1/n}^{\delta}\frac{A_{n,n-\tau}}{t\,\varepsilon(1/t)}\,dt\right)$$

$$= O\left(\frac{1}{\varepsilon(n)}\right) + O\left(\frac{1}{\varepsilon(n)}\right) + O(1) + O(1).$$

By virtue of (4.3) we have

$$I_2 = O(1)\ ,\ \text{as } n \to \infty \tag{6.3}$$

Since the method of summability is regular, we have

$$I_3 = O(1)\ ,\ \text{as } n \to \infty \tag{6.4}$$

by Riemann-Lebesgue theorem.

Combining the above results we obtain (6.1).

This completes the proof of the theorem.

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