Commun. Fac. Sci. Univ. Ank. Series A1
V. 45. pp. 85-95 (1996)

SOME EXTENSIONS OF GALLOP'S FORMULAS

FATMA TAŞDELEN and NURİ ÖZALP

Ankara University, Faculty of Sciences, 06100 Tandoğan-Ankara, TURKEY

(Received Nov. 1, 1996; Accepted Nov. 26, 1996)

ABSTRACT

In this study, some recurrence relations for a class of functions derived from Lorentzian metric, which can be solutions of a ultra-hyperbolic type linear P.D.E. in p+q dimensional space, are obtained.

1. INTRODUCTION

It is shown by E.G. Gallop [2] that in three dimensional space, the surface spherical harmonics defined by

$$P(a,b,c) = \frac{(-1)^n}{a!b!c!} r^{n+1} \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial y}\right)^b \left(\frac{\partial}{\partial z}\right)^c \frac{1}{r} \tag{1}$$

hold the recurrence relation

$$(a+1)(a+2)P(a+2,b,c)+(b+1)(b+2)P(a,b+2,c)+(c+1)(c+2)P(a,b,c+2) = 0 \tag{2}$$

and

$$\frac{\partial}{\partial x}\left[r^n P(a,b,c)\right] = -r^{n-1}[(a+1)P(a+1,b-2,c)+(a+1)P(a+1,b,c-2)-(2n-a)P(a-1,b,c)], \tag{3}$$

where $r^2=x^2+y^2+z^2$ and a, b, c are non-negative integers with n=a+b+c. It is well known that P(a,b,c) is a homogeneous function of zero degree and satisfies the Laplace equation [3].

In this study, we obtain some new forms of the formulas (2) and (3) by extending them to the functions which are the solutions of a P.D.E. of ultra-hyperbolic type in p+q dimension. We use the following notations:

$$L = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2} \tag{4}$$

$$s^2 = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 = |x|^2 - |y|^2 \tag{5}$$

2. SOME RECCURENCE RELATIONS FOR THE SOLUTIONS OF Lu=0

Let $a_1,\dots,a_p,b_1,\dots,b_q$ be non-negative integers and let $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = n$. Define the function P^* as

$$P^*(a_1,\dots,a_p,b_1,\dots,b_q) = \frac{1}{a_1!\dots a_p!b_1!\dots b_q!}\left(\frac{\partial}{\partial x_1}\right)^{a_1}\dots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\dots\left(\frac{\partial}{\partial y_q}\right)^{b_q} G(x_1,\dots,x_p,y_1,\dots,y_q), \tag{6}$$

where $G\in C^{n+2}(D)$ and $D\subset R^{p+q}$.

Lemma 1. If $L(G) = 0$, then $L(P^*) = 0$.

Proof. The operator L defined by (4) and the operator T defined by

$$T = \left(\frac{\partial}{\partial x_1}\right)^{a_1}\dots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\dots\left(\frac{\partial}{\partial y_q}\right)^{b_q} \tag{7}$$

are linear operators with constant coefficients and they hold the relation $LT(u) = TL(u)$. Thus, from (6),

$$\begin{aligned} L(P^*) &= L\left\{\frac{1}{a_1!\dots a_p!b_1!\dots b_q!} T(G)\right\} \\ &= \frac{1}{a_1!\dots a_p!b_1!\dots b_q!} L[T(G)] \\ &= \frac{1}{a_1!\dots a_p!b_1!\dots b_q!} T[L(G)] \end{aligned} \tag{8}$$

Since $L(G) = 0$, we get $L(P^*) = 0$ by (8).

Theorem 1. If $L(G) = 0$, then the function $P^*(a_1,\dots,a_p,b_1,\dots,b_q)$ as being a solution of $Lu = 0$, satisfies the recurrence formula

$$\sum_{i=1}^{p} (a_i+1)(a_i+2)P^*(a_1,\dots,a_i+2,\dots,a_p,b_1,\dots b_q) - \sum_{j=1}^{q} (b_j+1)(b_j+2)P^*(a_1,\dots,a_p,b_1,\dots,b_j+2,\dots b_q)=0. \tag{9}$$

Proof. By the equality (6), we have

$$\sum_{i=1}^{p} (a_i+1)(a_i+2)P^*(a_1,\dots,a_i+2,\dots,a_p,b_1,\dots b_q) - \sum_{j=1}^{q} (b_j+1)(b_j+2)P^*(a_1,\dots,a_p,b_1,\dots,b_j+2,\dots b_q)$$

$$= \sum_{i=1}^{p} \frac{(a_i+1)(a_i+2)}{a_1!\ldots(a_i+2)!\ldots a_p!b_1!\ldots b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_i}\right)^{a_i+2} \ldots \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} \ldots \left(\frac{\partial}{\partial y_q}\right)^{b_q} G$$

$$- \sum_{j=1}^{q} \frac{(b_j+1)(b_j+2)}{a_1!\ldots a_p!b_1!\ldots(b_j+2)!\ldots b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} \ldots \left(\frac{\partial}{\partial y_j}\right)^{b_j} \ldots \left(\frac{\partial}{\partial y_q}\right)^{b_q} G$$

$$= \frac{1}{a_1!\ldots a_p!b_1!\ldots b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} \ldots \left(\frac{\partial}{\partial y_q}\right)^{b_q} \left\{ \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2} \right\} G$$

$$= \frac{1}{a_1!\ldots a_p!b_1!\ldots b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} \ldots \left(\frac{\partial}{\partial y_q}\right)^{b_q} L(G).$$

Since $L(G) = 0$, the result follows.

Corollary 1. The function P_0^* defined by

$$P_0^*(a_1,\ldots,a_p,b_1,\ldots,b_q) = \frac{1}{a_1!\ldots a_p!b_1!\ldots b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} \ldots \left(\frac{\partial}{\partial y_q}\right)^{b_q} \frac{1}{s^{p+q-2}} \tag{10}$$

satisfies the recurrence formula (9). Here s is the Lorentzian distance defined by (5).

Proof. Since $L(s^{2-p-q}) = L\left(\frac{1}{s^{p+q-2}}\right) = 0$ (see [1]), by letting $G = \frac{1}{s^{p+q-2}}$ in Theorem 1, we get the result.

Theorem 2. Let $D \subset R^{p+q}$, $F \in C(D)$ and $G \in C^n(D)$. For $L(G) = 0$ the function P^{**} defined by

$$P^{**}(a_1,\ldots,a_p,b_1,\ldots,b_q) = \frac{F(x_1,\ldots,x_p,y_1,\ldots,y_q)}{a_1!\ldots a_p!b_1!\ldots b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} \ldots \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} \ldots \left(\frac{\partial}{\partial y_q}\right)^{b_q} G(x_1,\ldots,x_p,y_1,\ldots,y_q) \tag{11}$$

satisfies the recurrence relation

$$\sum_{i=1}^{p} (a_i+1)(a_i+2)P^{**}(a_1,\ldots,a_i+2,\ldots,a_p,b_1,\ldots b_q) - \sum_{j=1}^{q} (b_j+1)(b_j+2)P^{**}(a_1,\ldots,a_p,b_1,\ldots,b_j+2,\ldots b_q) = 0 \tag{12}$$

Proof. Multiplying both side of the equality (9) by $F(x_1,\ldots,x_p,y_1,\ldots,y_q)$ and by observing the relation $FP^* = P^{**}$ from (6) and (11), we get (12).

Remark. We note that the function P^{**} defined by (11) need not be a solution of the equation $Lu = 0$. But, if we choose, in (11),

$$F = (-1)^n s^{n+p+q-2} \quad \text{and} \quad G = \frac{1}{s^{p+q-2}},$$

then the resulted function

$$P(a_1,...,a_p,b_1,...,b_q) = \frac{(-1)^n s^{n+p+q-2}}{a_1!...a_p!b_1!...b_q!}\left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{p+q-2}} \tag{13}$$

is a zero-degree homogeneous solution of the ultra-hyperbolic equation Lu=0. By taking P instead of P** in (12), the function P as a special case of P** satisfies the recurrence relation

$$\sum_{i=1}^{p}(a_i+1)(a_i+2)P(a_1,...,a_i+2,...,a_p,b_1,...b_q) - \sum_{j=1}^{q}(b_j+1)(b_j+2)P(a_1,...,a_p,b_1,...,b_j+2,...b_q)=0. \tag{14}$$

The formula (14) is the extension of the recurrence relation (2), which is satisfied by the surface spherical harmonics as zero degree homogeneous solutions of Laplace equation to the similar type solutions of the equation Lu = 0.

3. EXTENSION OF (3)

In this section, some extensions of the recurrence formula (3) are obtained. Let us first give the following lemma.

Lemma 2. Let us define the function Q as

$$Q(a_1,...,a_p,b_1,...,b_q) = \left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}, \tag{15}$$

where s is the Lorentzian distance and ϕ is a real constant. Then we have

$$\begin{aligned} s^2Q(a_1,...,a_p,b_1,...,b_q) = &-\sum_{i=1}^{p}\left\{a_i(a_i-1)Q(a_1,...,a_i-2,...,a_p,b_1,...b_q)+2a_ix_iQ(a_1,...,a_i-1,...,a_p,b_1,...b_q)\right\} \\ &+\sum_{j=1}^{q}\left\{b_j(b_j-1)Q(a_1,...,a_p,b_1,...,b_j-2,...,b_q)+2b_jy_jQ(a_1,...,a_p,b_1,...,b_j-1,...,b_q)\right\} \\ &+(-\phi+2)\left\{(a_1-1)Q(a_1-2,...,a_p,b_1,...b_q)+x_1Q(a_1-1,...,a_p,b_1,...b_q)\right\}, \end{aligned} \tag{16}$$

where a_i, b_j (i=1,...,p,j=1,...q) are non-negative integers.

Proof. By the definition of s, we have

$$\frac{\partial}{\partial x_1}\left(s^{-\phi+2}\right) = (-\phi+2)\,\frac{x_1}{s^{\phi}}.$$

Applying the operator

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}$$

to the both sides of the above equality we get

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q} s^{-\phi+2} = (-\phi+2)\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{x_1}{s^{\phi}}. \tag{17}$$

On the other hand, rewriting the term $s^{-\phi+2}$ of the first side of the equality as $s^2\,\frac{1}{s^{\phi}}$ and taking the first derivative with respect to x_1 we obtain

$$\left(\frac{\partial}{\partial x_1}\right)\left(s^2\,\frac{1}{s^{\phi}}\right) = 2x_1\,\frac{1}{s^{\phi}} + s^2\,\frac{\partial}{\partial x_1}\left(\frac{1}{s^{\phi}}\right),$$

and hence the second derivative gives

$$\left(\frac{\partial}{\partial x_1}\right)^2\left(s^2\,\frac{1}{s^{\phi}}\right) = 2\,\frac{1}{s^{\phi}} + 4x_1\,\frac{\partial}{\partial x_1}\left(\frac{1}{s^{\phi}}\right) + s^2\,\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{s^{\phi}}\right)$$

By repeating derivation a_1 times, we obtain, by induction,

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(s^2\,\frac{1}{s^{\phi}}\right) = (a_1-1)a_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-2}\left(\frac{1}{s^{\phi}}\right) + 2a_1x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\left(\frac{1}{s^{\phi}}\right) + s^2\left(\frac{\partial}{\partial x_1}\right)^{a_1}\frac{1}{s^{\phi}}$$

Similarly, in the last expression, taking the derivatives a_2 times with respect to x_2 gives us

$$\left(\frac{\partial}{\partial x_1}\right)^{a_2}\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(s^2\,\frac{1}{s^{\phi}}\right) = \left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\left(s^2\,\frac{1}{s^{\phi}}\right)$$

$$= (a_1-1)a_1\left(\frac{\partial}{\partial x_2}\right)^{a_2}\left(\frac{\partial}{\partial x_1}\right)^{a_1-2}\left(\frac{1}{s^{\phi}}\right) + 2a_1x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\left(\frac{\partial}{\partial x_1}\right)^{a_2}\left(\frac{1}{s^{\phi}}\right)$$

$$+ (a_2-1)a_2\left(\frac{\partial}{\partial x_2}\right)^{a_2-2}\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{1}{s^{\phi}}\right) + 2a_2x_2\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2-1}\left(\frac{1}{s^{\phi}}\right) + s^2\left(\frac{\partial}{\partial x_2}\right)^{a_2}\left(\frac{\partial}{\partial x_1}\right)^{a_1}\frac{1}{s^{\phi}}.$$

Proceeding in this manner, taking the derivatives with respect to $x_3,\ldots,x_p$ respectively $a_3,\ldots,a_p$ times, finally we obtain

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(s^2\,\frac{1}{s^\phi}\right) = (a_1-1)a_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-2}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi}$$

$$+\ 2a_1x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi} + (a_2-1)a_2\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2-2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi}$$

$$+\ 2a_2x_2\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi} + (a_3-1)a_3\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\left(\frac{\partial}{\partial x_3}\right)^{a_3-2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi}$$

$$+\ 2a_3x_3\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\left(\frac{\partial}{\partial x_3}\right)^{a_3-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi} +\ldots+ (a_p-1)a_p\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p-2}\frac{1}{s^\phi}$$

$$+\ 2a_px_p\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p-1}\frac{1}{s^\phi} + s^2\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{s^\phi}$$

Now, by applying the operators $\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}$ successively to both sides of the last expression, we get

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\left(s^2\,\frac{1}{s^\phi}\right)$$

$$= (a_1-1)a_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-2}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}$$

$$+\ 2a_1x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi} + \ldots$$

$$+\ (a_p-1)a_p\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p-2}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}$$

$$+\ 2a_px_p\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p-1}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}$$

$$-\ (b_1-1)b_1\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1-2}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}$$

$$-\ 2b_1y_1\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1-1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}$$

$$-\ \ldots\ -\ (b_q-1)b_q\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q-2}\frac{1}{s^\phi}$$

$$- 2b_q y_q \left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q-1}\frac{1}{s^\phi}$$
$$+ s^2\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q-1}\frac{1}{s^\phi}.$$

By using the definition of Q, we can rewrite the right hand side of the last equality in terms of Q as follows:

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\left(s^2\,\frac{1}{s^\phi}\right)$$
$$= \sum_{i=1}^{p}\left\{a_i(a_i-1)Q(a_1,\dots,a_i-2,\dots,a_p,b_1,\dots,b_q)+2a_ix_iQ(a_1,\dots,a_i-1,\dots,a_p,b_1,\dots,b_q)\right\}$$
$$- \sum_{j=1}^{q}\left\{b_j(b_j-1)Q(a_1,\dots,a_p,b_1,\dots,b_j-2,\dots,b_q)+2b_jy_jQ(a_1,\dots,a_p,b_1,\dots,b_j-1,\dots,b_q)\right\}$$
$$+ s^2Q(a_1,\dots,a_p,b_1,\dots,b_q) \tag{18}$$

Similarly, applying the operator $\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}$ to $\frac{x_1}{s^\phi}$, the right hand side of the equality (17) takes the form

$$(-\phi+2)\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{x_1}{s^\phi} =$$
$$(-\phi+2)\left\{(a_1-1)\left(\frac{\partial}{\partial x_1}\right)^{a_1-2}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}+x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^\phi}\right\}.$$

Again by using the definition of Q, we get

$$(-\phi+2)\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{x_1}{s^\phi}$$
$$= (-\phi+2)\left\{(a_1-1)Q(a_1-2,\dots,a_p,b_1,\dots,b_q)+x_1Q(a_1-1,\dots,a_p,b_1,\dots,b_q)\right\}. \tag{19}$$

Hence in view of (17) (18) and (19) we obtain (16).

Theorem 3. Q as being in Lemma 2, we have

$$\frac{\partial}{\partial x_1}\left[s^\alpha Q(a_1,\dots,a_p,b_1,\dots,b_q)\right] = s^{\alpha-2}\left\{(-\phi+1-2n-a_1)a_1Q(a_1-1,\dots,a_p,b_1,\dots,b_q)\right\}$$
$$+ (-2n-\phi+\alpha)x_1Q(a_1,\dots,a_p,b_1,\dots,b_q)+\sum_{i=2}^{p}a_i(a_i-1)Q(a_1+1,\dots,a_i-2,\dots,a_p,b_1,\dots,b_q)$$

$$- \sum_{j=1}^{q} b_j(b_j-1)Q(a_1+1,...,a_p,b_1,...,b_j-2,...,b_q), \tag{20}$$

where α and ϕ are real constants, $n = \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j$ and s is Lorentzian distance defined in (5).

Proof. Using (15) in (20) and writing the derivative with respect to x_1 explicitly we have

$$\frac{\partial}{\partial x_1}\left[s^{\alpha}Q(a_1,...,a_p,b_1,...,b_q)\right] = \frac{\partial}{\partial x_1}\left[s^{\alpha}\left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}\right]$$

$$= s^{\alpha-2}\left[s^2\left(\frac{\partial}{\partial x_1}\right)^{a_1+1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}+\alpha x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}\right] \tag{21}$$

On the other hand in (16) replacing a_1 by a_1+1 and using this value in (21), after simplifications wet get

$$\frac{\partial}{\partial x_1}\left[s^{\alpha}Q(a_1,...,a_p,b_1,...,b_q)\right]$$
$$= s^{\alpha-2}\Big[(-\phi-a_1+1)a_1Q(a_1-1,...,a_p,b_1,...,b_q)+(-2a_1-\phi+\alpha)x_1Q(a_1,...,a_p,b_1,...,b_q)$$
$$- \sum_{i=2}^{p}\left\{a_i(a_i-1)Q(a_1+1,...,a_i-2,...,a_p,b_1,...,b_q)+2a_ix_iQ(a_1+1,...,a_i-1,...,a_p,b_1,...,b_q)\right\}$$
$$- \sum_{j=1}^{q}\left\{b_j(b_j-1)Q(a_1+1,...,a_p,b_1,...,b_j-2,...,b_q)+2b_jy_jQ(a_1+1,...,a_p,b_1,...,b_j-1,...,b_q)\right\}\Big] \tag{22}$$

Next, for i = 1,2,...,p since $\left(x_1\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_1}\right)\frac{1}{s^{\phi}} = 0$, applying both sides the operator

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_i}\right)^{a_i-1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q},$$

we get, for i=2,...,p,

$$a_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}+x_1\left(\frac{\partial}{\partial x_1}\right)^{a_1-1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}$$
$$- (a_1-1)\left(\frac{\partial}{\partial x_1}\right)^{a_1+1}...\left(\frac{\partial}{\partial x_i}\right)^{a_i-2}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}$$

$$- x_i\left(\frac{\partial}{\partial x_1}\right)^{a_1+1}\cdots\left(\frac{\partial}{\partial x_i}\right)^{a_i-1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}} = 0.$$

By using the definition of Q in the above equality we obtain

$$a_1Q(a_1-1,\ldots,a_p,b_1,\ldots,b_q)+x_1Q(a_1,\ldots,a_p,b_1,\ldots,b_q)-(a_i-1)Q(a_1+1,\ldots,a_i-2,\ldots,a_p,b_1,\ldots,b_q)$$
$$-x_iQ(a_1+1,\ldots,a_i-1,\ldots,a_p,b_1,\ldots,b_q) = 0,\ i = 2,\ldots,p \tag{23}$$

Similarly, applying the operator

$$\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_j}\right)^{b_j-1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}$$

to both sides of $\left(x_1\frac{\partial}{\partial y_j} + y_j\frac{\partial}{\partial x_1}\right)\frac{1}{s^{\phi}} = 0$ (j=1,2,...,q), and again using the definition of Q, we get

$$a_1Q(a_1-1,\ldots,a_p,b_1,\ldots,b_q)+x_1Q(a_1,\ldots,a_p,b_1,\ldots,b_q)$$
$$+(b_j-1)Q(a_1+1,\ldots,a_p,b_1,\ldots,b_j-2,\ldots,b_q)+y_jQ(a_1+1,\ldots,a_p,b_j,\ldots,b_j-1,\ldots,b_q) = 0. \tag{24}$$

Multiplying (23) and (24) respectively with $-2a_i$, (i=2,...,p) and $-2b_j$, (j=1,2,...,q) and adding them side by side, and then comparing the result with the right side of (22), we obtain

$$\frac{\partial}{\partial x_1}\left[s^{\alpha}\left(\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{\phi}}\right]$$
$$= s^{\alpha-2}\Big\{(-\phi+1-2n-a_1)a_1Q(a_1-1,\ldots,a_p,b_1,\ldots,b_q)$$
$$+(-2n-\phi+\alpha)x_1Q(a_1,\ldots,a_p,b_1,\ldots,b_q) + \sum_{i=2}^{p} a_i(a_i-1)Q(a_1+1,\ldots,a_i-2,\ldots,a_p,b_1,\ldots,b_q)$$
$$- \sum_{j=1}^{q} b_j(b_j-1)Q(a_1+1,\ldots,a_p,b_1,\ldots,b_j-2,\ldots,b_q)\Big\}$$

Corollary 2. Let s and P be as defined in (5) and (13). Then

$$\frac{\partial}{\partial x_1}\left[s^nP(a_1,\ldots,a_p,b_1,\ldots,b_q)\right] = -s^{n-1}\Big\{\sum_{i=2}^{p}(a_1+1)P(a_1+1,\ldots,a_i-2,\ldots,a_p,b_1,\ldots,b_q)$$
$$-\sum_{j=1}^{q}(a_1+1)P(a_1+1,\ldots,a_p,b_1,\ldots,b_j-2,\ldots,b_q) - (2n+p+q-3-a_1)P(a_1-1,\ldots,a_p,b_1,\ldots,b_q)\Big\} \tag{25}$$

Proof. Let us take $\alpha = 2n+p+q-2$, and $\phi = p+q-2$ in (20). Then

$$\frac{\partial}{\partial x_1}\left[s^{2n+p+q-2}Q(a_1,...,a_p,b_1,...,b_q)\right]$$
$$= s^{2n+p+q-4}\Big\{(-p-q+3-2n-a_1)a_1Q(a_1-1,...,a_p,b_1,...,b_q)$$
$$+ (-2n-(p+q+2)+2n+p+q-2)x_1Q(a_1,...,a_p,b_1,...,b_q)+ \sum_{i=2}^{p} a_i(a_i-1)Q(a_1+1,...,a_i-2,...,a_p,b_1,...,b_q)$$
$$- \sum_{j=1}^{q} b_j(b_j-1)Q(a_1+1,...,a_p,b_1,...,b_q)\Big\}.$$

We obtain (25), by multiplying the last equality by $\dfrac{(-1)^n}{a_1!...a_p!b_1!...b_q!}$ and taking the equality

$$P(a_1,...,a_p,b_1,...,b_q) = (-1)^n \frac{s^{n+p+q-2}}{a_1!...a_p!b_1!...b_q!} Q(a_1,...,a_p,b_1,...,b_q)$$

into account.

Remark. Using $\dfrac{\partial}{\partial y_1}$ instead of $\dfrac{\partial}{\partial x_1}$ replaces Theorem 2 and Corollary 1 with followings:

Theorem 4.

$$\frac{\partial}{\partial y_1}\left[s^{\alpha}Q(a_1,...,a_p,b_1,...,b_q)\right] = s^{\alpha-2}\Big[(\phi-1+2n-b_1)b_1Q(a_1,...,a_p,b_1-1,...,b_q)$$
$$+ (\phi-\alpha+2n)y_1Q(a_1,...,a_p,b_1,...,b_q)+ \sum_{i=1}^{p} a_i(a_i-1)Q(a_1,...,a_i-2,...,a_p,b_1+1,...,b_q)$$
$$- \sum_{j=2}^{q} b_j(b_j-1)Q(a_1,...,a_p,b_1+1,...,b_j-2,...,b_q)\Big].$$

Corollary 3.

$$\frac{\partial}{\partial y_1}\left[s^{n}P(a_1,...,a_p,b_1,...,b_q)\right] = -s^{n-1}\Big[\sum_{i=1}^{p} (b_1+1)P(a_1,...,a_i-2,...,a_p,b_1+1,...,b_q)$$
$$- \sum_{j=2}^{q} (b_1+1)P(a_1,...,a_p,b_1+1,...,b_j-2,...,b_q) + (2n+p+q-3-b_1)P(a_1,...,a_p,b_1-1,...,b_q)\Big],$$

where $n = \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j$ and s and $P(a_1,...,a_p,b_1,...,b_q)$ are defined as in (5) and (13) respectively.

Acknowledgement: We are thankful to Professor Abdullah Altın for his valuable suggestions.

REFERENCES

[1] ALTIN, A., Some Expansion Formulas for a Class of Singular Partial Differential Equations, Proc. Am. Math. Soc. 85, No. 1 (1982), 42-46.

[2] GALLOP, E.G., The Differentiation of Spherical Harmonics, Proc. Lond. Math. Soc. (1), Vol. XXVIII, (1896).

[3] HOBSON, E.W., The Theory of Spherical and Ellipsoidal Harmonics, Chelsea Pub. Co., New York., 1965.