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HOMOGENEOUS SOLUTIONS FOR A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

We recall that a spherical harmonic is a homegeneous function of x, y, z of certain degree n which satisfies Laplace equation. Thus, if V(x,y,z) is such a function of degree λ, then $xV_x+yV_y+zV_z = \lambda V(x,y,z)$, and $\Delta V \equiv V_{xx}+V_{yy}+V_{zz} = 0$. An important result in the theory of harmonic functions is that any harmonic function can be expressed in a series involving the spherical harmonics.

In this paper we shall study homogeneous functions which satisfy the general elliptic-ultrahyperbolic partial differential equation

$$L(u) = \sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i}\frac{\partial u}{\partial x_i}\right) \pm \sum_{j=1}^{s}\left(\frac{\partial^2 u}{\partial y_j^2} + \frac{\beta_j}{y_j}\frac{\partial u}{\partial y_j}\right) + \frac{\gamma}{r^2}u = 0 \qquad (1)$$

where α_i $(1\le i\le n)$, β_j $(1\le j\le s)$ and γ are real parameters and

$$r^2 = \sum_{i=1}^{n} x_i^2 \pm \sum_{j=1}^{s} y_j^2 = |x|^2 \pm |y|^2 \qquad (2)$$

The domain of the operator L is the set of all real valued functions u(x,y) of class $C^2(D)$, where $x = (x_1,...,x_n)$ and $y = (y_1,...,y_s)$ denote points in R^n and R^s, respectively, and D is a regularity domain of u in R^{n+s}. Clearly the equation (1) includes some of the well known classical equations of mathematical physics such as the Laplace equation, the wave equation and EPD and GASPT equations [1-5]. The equation (1) was considered by Altın [2] for which some expansion formulas for solutions of the iterated forms of the equation were given.

2. HOMOGENEOUS SOLUTIONS

We first give some properties of the operator L. In [2] the following two properties of L are given.

(i) For any real parameter m,

$$L(r^m) = [m(m + \phi) + \gamma]\, r^{m-2} \tag{3}$$

where

$$\phi = n + s - 2 + \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{s} \beta_j \tag{4}$$

(ii) If $u,v \in C^2(D)$ are any two functions, then the operator L satisfies the relation

$$L(uv) = uL(v) + vL(u) - \frac{\gamma}{r^2} uv + 2\left(\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \pm \sum_{j=1}^{s} \frac{\partial u}{\partial y_j}\frac{\partial v}{\partial y_j}\right) \tag{5}$$

In (5), taking $u = r^m$ and $v = V_\lambda(x,y)$ which is a homogeneous function of degree λ, we then obtain the formula

$$L(r^m V_\lambda) = m(m + 2\lambda + \phi) r^{m-2} V_\lambda + r^m L(V_\lambda) \tag{6}$$

This formula will play an important role in finding homogeneous solutions of our equation (1). By making use of the formula (6) we shall prove the following theorem.

Theorem 1. Let $V_\lambda(x,y) \in C^\infty(D)$ be any homogeneous function of degree λ. If $2\lambda + \phi$ is not a positive even number, then the function

$$W_\lambda(x,y) = \left\{1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda,\phi) r^{2q} L^q\right\} V_\lambda(x,y) \tag{7}$$

where

$$a_q(\lambda,\phi) = \frac{1}{2.4\ldots(2q)(2\lambda+\phi-2)(2\lambda+\phi-4)\ldots(2\lambda+\phi-2q)} \tag{8}$$

and

$$L^{q+1} = L(L^q) \text{ for } q = 1,2,\ldots$$

is a homogeneous solution of degree λ of the equation (1).

Proof. Using the properties of homogeneous functions and the definition of L, we can see that $L^q(V_\lambda(x,y))$ is a homogeneous function of degree λ-2q for any positive integer q. Since the factor r^{2q} is homogeneous of degree 2q, each term $r^{2q}L^q(V_\lambda)$ of (7) is again a homogeneous function of degree λ, and therefore the limit function $W_\lambda(x,y)$ will be also a homogeneous function of the same degree λ. Hence, by the relation (6) we have

$$L[r^{2q}L^q(V_\lambda)] = 2q[2q + 2(\lambda-2q) + \phi]r^{2q-2}L^q(V_\lambda) + r^{2q}L^{q+1}(V_\lambda) \quad (9)$$
$$= 2q(2\lambda + \phi - 2q)r^{2q-2}L^q(V_\lambda) + r^{2q}L^{q+1}(V_\lambda)$$

Now let us apply the operator L on both sides of (7) and use the formula (9). We obtain

$$L(W_\lambda) = L(V_\lambda) + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda,\phi) r^{2q} L\left[r^{2q}L^q(V_\lambda)\right]$$

$$= L(V_\lambda) + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda,\phi) \left\{(2q)(2\lambda+\phi-2q)r^{2q-2}L^q(V_\lambda)+r^{2q}L^{q+1}(V_\lambda)\right\}$$
$$= L(V_\lambda) - a_1(\lambda,\phi)2(2\lambda+\phi-2)L(V_\lambda)$$
$$+ \sum_{q=2}^{\infty} (-1)^q \left\{(2q)(2\lambda+\phi-2q)a_q(\lambda,\phi)-a_{q-1}(\lambda,\phi)\right\}r^{2q-2}L^q(V_\lambda).$$

On the other hand from the definition of $a_q(\lambda,\phi)$, it is clear that

$$2(2\lambda + \phi - 2)a_1(\lambda,\phi) = 1$$

and

$$2q(2\lambda + \phi - 2q)a_q(\lambda,\phi) = a_{q-1}(\lambda,\phi) \quad ; \quad q = 2, 3, \ldots$$

Therefore, $L(W_\lambda) \equiv 0$, which proves our theorem.

3. SOLUTIONS FOR THE ITERATED EQUATION $L^p u = 0$.

First we shall prove the following lemma.

Lemma 1. Let $V_\lambda(x,y)$ be any homogeneous function of degree λ. Then for any positive integer p and for any real number m

$$L^p(r^m V_\lambda) = \sum_{k=0}^{p} c(p,k)\, r^{m-2k} L^{p-k}(V_\lambda) \quad (10)$$

where

$$L^\circ(V_\lambda) = V_\lambda,\ c(0,0) = c(p,0) = 1,\ c(p,1) = mp(m+2-2p+2\lambda+\phi),$$

$$c(p,k) = c(p-1,k)+(m+2-2k)(m+2-4p+2k+2\lambda+\phi)c(p-1,k-1);\ k = 1,...,p-1$$

$$c(p,p) = \prod_{j=0}^{p-1} (m-2j)(m-2j+2\lambda+\phi) \text{ and } c(p,k) = 0 \text{ for } k > p.$$

Proof. Applying the operator L on both sides of the formula (6) and noting that $L(V_\lambda)$ is a homogeneous function of degree λ-2, we have

$$\begin{aligned} L^2(r^m V_\lambda) &= m(m+2\lambda+\phi)\{(m-2)(m-2+2\lambda+\phi)r^{m-4}V_\lambda+r^{m-2}L(V_\lambda)\} \\ &\quad +m[m+2(\lambda-2)+\phi]r^{m-2}L(V_\lambda)+r^m L^2(V_\lambda) \\ &= r^m L^2(V_\lambda)+2m(m-2+2\lambda+\phi)r^{m-2}L(V_\lambda) \\ &\quad +m(m-2)(m+2\lambda+\phi)(m-2+2\lambda+\phi)r^{m-4}V_\lambda \\ &= c(2,0)r^m L^2(V_\lambda)+c(2,1)r^{m-2}L(V_\lambda)+c(2,2)r^{m-4}V_\lambda \end{aligned}$$

Hence by induction we obtain the formula (10). We note that if V_λ is a solution of the equation $L(u) = 0$, then our formula (10) takes the form

$$\begin{aligned} L^p(r^m V_\lambda) &= c(p,p)r^{m-2p}V_\lambda \\ &= r^{m-2p} \prod_{j=0}^{p-1} (m-2j)(m-2j+2\lambda+\phi)V_\lambda \end{aligned} \tag{11}$$

By making use of Lemma 1 we shall now establish the following theorem.

Theorem 2. Let $V_{\lambda_j}(x,y)$ be any p homogeneous integral functions of degree λ_j for $j = 0,1,...,p-1$, respectively. Then the functions

$$\text{(a) } u_1 = \sum_{j=0}^{p-1} r^{2j} \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi) r^{2q} L^q \right\} V_{\lambda_j}(x,y)$$

and

$$\text{(b) } u_2 = \sum_{j=0}^{p-1} r^{2j-2\lambda_j-\phi} \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi) r^{2q} L^q \right\} V_{\lambda_j}(x,y)$$

satisfy the iterated equation $L^p(u) = 0$. Here L, r, ϕ and $a_q(\lambda,\phi)$ are defined by (1), (2), (4) and (8) respectively.

Proof. Since V_{λ_j} is a homogeneous integral function of degree λ_j, by Theorem 1

$$W_{\lambda_j}(x,y) = \left\{1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi) r^{2q} L^q\right\} V_{\lambda_j}(x,y)$$

is a homogeneous solution of degree λ_j of the equation $L(u) = 0$. Therefore, from the formula (11) of Lemma 1, we have

$$L^p(r^m W_{\lambda_j}) = r^{m-2p} \prod_{j=0}^{p-1} (m-2j)(m-2j+2\lambda_j+\phi) W_{\lambda_j} \tag{12}$$

Thus, by (12), for $j = 0,1,\ldots,p-1$, we have

$$L^p[r^{2j} W_{\lambda_j}] = 0 \text{ and } L^p[r^{2j-2\lambda_j-\phi} W_{\lambda_j}] = 0$$

Hence, by the principle of superposition, it follows that u_1 and u_2 both satisfy the equation $L^p(u) = 0$.

We notice that the solution u_1 is a special case of Almansi's expansion for the equation (1) and the solution u_2 is a homogeneous function expansion for the same equation (1). Both of them were obtained in [2] using a different method.

4. SOME REMARKS

(i) Suppose V_λ is a homogeneous integral function of degree λ such that $2\lambda+\phi$ is not a positive even number. Since the function

$$W_\lambda(x,y) = \left\{1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi) r^{2q} L^q\right\} V_\lambda(x,y)$$

is a solution of the equation (1) and since Kelvin principle is valid for the same equation [2,3], the function

$$u(x,y) = r^{-\phi} W_\lambda(\xi,\eta)$$

is also a solution of the same equation (1). Here $\xi = (\xi_1,\ldots,\xi_n)$, $\eta = (\eta_i,\ldots,\eta_s)$ and $\xi_i = x_i/r^2$, $(1 \le i \le n)$, $\eta_j = y_j/r^2$ $(1 \le j \le s)$ and r and ϕ are defined before by (2) and (4).

(ii) In [2] it was shown that if $V_\lambda(x,y)$ is a homogeneous solution of degree λ of the equation (1), then

$$L^p[r^m V_\lambda(\xi,\eta)] = r^{m-2p} \prod_{j=0}^{p-1} (m-2j+\phi)(m-2j-2\lambda) V_\lambda(\xi,\eta) \tag{13}$$

Using Theorem 2 and the formula (13), we can give two more solution for the iterated equation $L^p u = 0$.

Let $V_{\lambda_j}(x,y)$ be any p homogeneous integral function of degree λ_j for $j = 0,1,...,p-1$ and define $W_{\lambda_j}(x,y)$ as

$$W_{\lambda_j}(x,y) = \left\{ 1 + \sum_{q=1}^{\infty} (-1)^q a_q(\lambda_j,\phi) r^{2q} L^q \right\} V_{\lambda_j}(x,y) \ , \ j = 0,...,p-1$$

which are homogeneous solution of degree λ_j of the equation (1). From (13) we can say that

$$u_3(x,y) = \sum_{j=0}^{p-1} r^{2j-\phi} W_{\lambda_j}(\xi,\eta)$$

and

$$u_4(x,y) = \sum_{j=0}^{p-1} r^{2(j+\lambda_j)} W_{\lambda_j}(\xi,\eta)$$

are also solutions of the iterated equation $L^p(u) = 0$.

(iii) It is clear that by a simple linear transformation, Theorem 1 can be readily extended to the more general equation of the form

$$L_1(u) = \sum_{i=1}^{n} \left(a_i^2 \frac{\partial^2 u}{\partial t_i^2} + \frac{\alpha_i}{t_i - t_i^0} \frac{\partial u}{\partial t_i} \right) \pm \sum_{j=1}^{s} \left(b_j^2 \frac{\partial^2 u}{\partial z_j^2} + \frac{\beta_j}{z_j - z_j^0} \frac{\partial u}{\partial z_j} \right) + \frac{\gamma}{r_1^2} u = 0 \quad (14)$$

where a_i, b_j, α_i, β_j are real parameters ($a_i \neq 0$, $b_j \neq 0$), $t^0 = (t_1^0,...,t_n^0)$ and $z^0 = (z_1^0,...,z_s^0)$ are fixed points in $\mathbb{R}^n$ and $\mathbb{R}^s$, respectively, and r_1 denoted by

$$r_1^2 = \sum_{i=1}^{n} \left(\frac{t_i - t_i^0}{a_i} \right)^2 \pm \sum_{j=1}^{s} \left(\frac{z_j - z_j^0}{b_j} \right)^2$$

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