ON HOLDITCH AND LIOUVILLE THEOREMS

by

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SUMMARY

In this work, we show the connection between the Liouville theorem in mechanics and the Holditch theorem in geometry.

1. INTRODUCTION

There is a similarity between the theorems of Liouville and Holditch. We recall first the well known Holditch Theorem of geometry [1].

Let γ be a smooth closed curve (oval) in Euclidean Plane and l be a line segment of constant length which is less then the diameter of γ with the moving end points attached to γ. If X is a point on l with distances a and b to the end points of l, during the motion of l the area between the orbit γ_x of X and γ can be given by πab. In other words, the area is independent of the motion (i.e. independent of the choice of γ).

If the area bounded by the closed curve γ_x drawn by the point

$X = (x_1, x_2)$ on the plane O', e_1', e_2' is denoted by f_x, letting $\overrightarrow{X} = \overrightarrow{OX}$

and $\overrightarrow{X'} = \overrightarrow{O'X} = (x'_1, x'_2)$, we can write

$$f_x = \tfrac{1}{2} \int_{\gamma_x} < \overrightarrow{X'}, \overrightarrow{dX'} >$$
$$= \tfrac{1}{2} \int_{\gamma_x} (x'_1 dx'_2 - x'_2 dx'_1).$$

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Since the metric here is the Euclidean metric and the motion is Euclidean, the invariant $f - f_\gamma$, is one of the invariants of the Euclidean space, where f is the area bounded by the curve γ.

Now we ask the following question:

Is there an invariant (or invariants) similar to the one given above if we replace the Euclidean plane with a Riemannian manifold?

The answer is affirmative, in our another study [2] we show the existence of such invariants. The main aim of this work is to show that Liouville theorem in mechanics is an analogue of Holditch Theorem of geometry which is given above.

2. HAMILTONIAN AND GEODESICS

Let M be a Riemannian manifold of dimension n. It is clear that there can be no integral invariants of M if we do not impose any condition on M. We present some invariants imposing some conditions on M in [2].

Meanwhile the Holditch theorem of Euclidiean plane has a connection with well–known Liouville Theorem of mechanic. So it may be wise to have a look at Liouville Theorem and its connection with Holdicth Theorem.

Consider the Hamiltonian function H defined by

$$H : M \times TM(X) \longrightarrow R$$

$$(X, \vec{\xi}) \longrightarrow H(X, \vec{\xi}) = g^{ij}(X)\, \xi_i \xi_j$$

where $[g^{ij}]$ denotes the inverse matrix of $[g_{ij}]$ and the corresponding Riemannian metric [3] is given by

$$ds^2 = g_{ij}(X)\, dx^i dx^j$$

where $\vec{\xi}$ is the velocity vector at the point X of the orbit drawn by the point X, x^i is the ith coordinate function and ξ_i is the ith component of $\vec{\xi}$, with respect to the coordinate basis $\dfrac{\partial}{\partial x^i}$. Let ds be the arc–length element of the orbit drawn by the point X and assume that we load a mass on X. Suppose that the motion of these masses are independent of time but depend on the points.

As is well–known from mechanics [4]

$$
\left.
\begin{aligned}
\frac{\partial x^i}{\partial s} &= \frac{\partial H}{\partial \xi_i} \\[2mm]
\frac{\partial \xi_i}{\partial s} &= - \frac{\partial H}{\partial x_i}
\end{aligned}
\right\} 1 \leq i \leq n \tag{1}
$$

i.e. the orbit curve drawn by the point X is the solution of the system (1). This means that if the initial point and the velocity vector at that point are given, then the orbit curve is given, that is the **motion** is given.

The first n–equations $\left(\dfrac{dx^i}{ds} = \dfrac{\partial H}{\partial \xi_i}, \ 1 \leq i \leq n \right)$ in system (1)

express how **the** orbit of the point X in M changes. Similarly the second

n–equations $\left(\dfrac{d\xi_i}{ds} = - \dfrac{\partial H}{\partial x_i}, \ 1 \leq i \leq n \right)$ in system (1) express how

the velocity vector of this orbit changes. The system (1) is known as the Hamilton–Jacobi system.

An important property concerning geodesic curves of Hamilton–Jacobi system is rather interesting for geometry.

Theorem 2.1. Let M be a Riemannian manifold of dimension n and X be a point moving en M with respect to (1) Hamilton–Jacobi system. Then the orbit of X is a geodesic relative to the metric

$$
ds^2 = g_{ij}(X) \, dx^i dx^j.
$$

Proof: If s is the arc–length of the geodesic path as the natural parameter, the equality

$$
H(X, \vec{\xi}) = g^{ij}(X) \, \xi_i \xi_j - 1 = 0 \tag{2}
$$

can be interpreted as hypersurfaces of level 1 of the set of the vectors of length 1.

On the other hand, if we write the Hamilton–Jacobi system for this hypersurface we obtain,

$$
\frac{dx^i}{ds} = \frac{\partial H}{\partial \xi_i} = g^{ij}(X) \, \xi_j \Rightarrow \frac{dx^i}{ds} = g^{ij}(X) \, \xi_j \tag{3}
$$

$$\frac{d\xi_i}{ds} = \frac{\partial H}{\partial x^i} = -\frac{\partial g^{ij}}{\partial x_i}\xi_i\xi_j \Rightarrow \frac{d\xi_i}{ds} = -\frac{\partial g^{ij}}{\partial x^i}\xi_i\xi_j. \quad (4)$$

From (3) we have, $\xi_i = g_{ij}\dfrac{dx^j}{ds}$ and if we substitute this in (2)

for $g^{ij}(X)\ \xi_i\xi_j = 1$ we get

$$g^{ij}\ g_{ij}\ \frac{dx^j}{ds}\cdot g_{ij}\ \frac{dx^i}{ds} = 1$$

or

$$\delta_{ij}\ \frac{dx^j}{ds}\cdot g_{ij}\ \frac{dx^i}{ds} = 1$$

or

$$g_{ij}\ \frac{dx^i}{ds}\cdot\frac{dx^j}{ds} = 1$$

which gives us /

$$g_{ij}dx^idx^j = ds^2$$

from which the arc–length element of the curve, in concern, is obtained as

$$ds = (g_{ij}(X)\ dx^idx^j).^{1/2} \quad (5)$$

For a curve $L(X, X^0)$ which has an arc–length element of form (5) to be a geodesic, it is necessary the functional

$$\tau(L) = \int_{L(X,\ X^0)} [g_{ij}(X)dx^idx^j]^{1/2} \quad (6)$$

attains a minimum on this curve.

It is clear that the geodesic curves of the metric (5) are the solutions of Euler–Largrange system which is corresponding to functional (6), see [6]. If the arc–length parameter s is the natural parameter then the Euler–Lagrange system is the system (3) and (4), where

$$\xi_i = g_{ij}\ \frac{dx^j}{ds}\ .$$

The solutions of (1) are called the orbits which are corresponding to $H(X, \vec{\xi})$. Here we assume that the orbits (geodesics) of $H(X, \vec{\xi})$ are regular, i.e., there is only one geodesic passing through any two points of M.

Besides, the masses loaded to $X \in M$ are assumed to be independent of each other and they are not effected by the source field in which they are or they are not sensitive to the effects around them.

3. LIOUVILLE THEOREM

Definition 3.1. Let M be an n–Riemannian manifold and $\Omega \subset M$ be a domain on M. Let $X \in \Omega$, $\vec{\xi} \in T\Omega(X)$, $W \subset \Omega \times T\Omega(X)$ and $U(X, \vec{\xi})$ denotes the density of the masses included in the volume element

$$dW = dV^{2n} = \det (g_{ij}) \, d\xi_1 \wedge \ldots \wedge d\xi_n \wedge dx^1 \wedge \ldots \wedge dx^n$$

then the number of the little masses in W is given by [5]

$$\int_W U(X, \vec{\xi}) \, dV^{2n}.$$

If there is no source in W effecting the little masses as is known from mechanics [4], [5]

$$dU = \sum_{i=1}^{n} \xi_i \, \frac{\partial U}{\partial x^i} - \sum_{i'j'k=1}^{n} \Gamma^i_{jk} \, \xi_j \xi_k \, \frac{\partial U}{\partial \xi_i} = 0. \tag{7}$$

We can write (7) simply in the form

$$dU = \xi_i U_{\times i} - \Gamma^i_{jk} \, \xi_j \xi_k U \xi_i = 0 \tag{8}$$

where dU is the exact differential of the density function with respect to the metric $g = [g_{ij}]$ along the corresponding geodesic path.

Now we explain the physical meaning of (8):

All corresponding orbits to the Hamiltonian H at all points of a certain set at M are called the pipe of rays.

Theorem 3.1. (Liouville Theorem): The number of the little masses at each section of two parallel sections of pipe of rays of an

n–dimensional Riemannian manifold is independent from these sections [5].

The body bounded by these sections is the domain W in $\Omega \times TM$. If there are sources in W of density F, the equation (8) turns into

$$dU = \xi_i U_{\times i} - \Gamma^i{}_{jk} \xi_j \xi_k U_{\xi i} = F \tag{9}$$

see [4] and [5].

Theorem 3.2. (Generalized Liouville Theorem). The following connection exists between the domain W and the density of sources F:

$$\int_W F dW = \int_{F_2} U dF_2 - \int_{F_1} U dF_1. \tag{10}$$
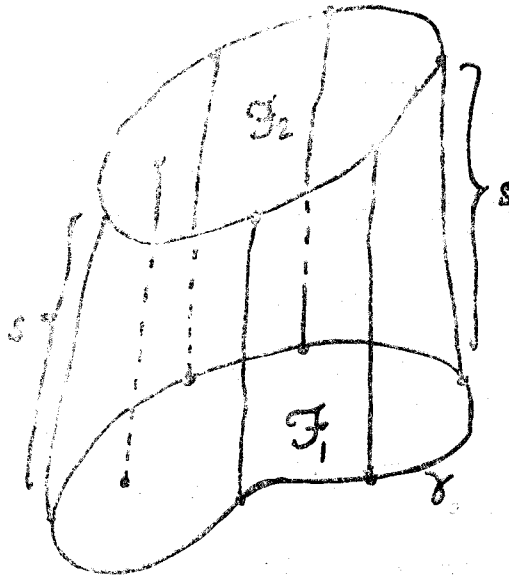
(see the figure)

Proof At first we consider the case corresponding to $F = 0$. In this case from (10) we obtain

$$\int_{F_2} U dF_2 = \int_{F_1} U dF_1.$$

This is the Liouville theorem which means that the number of the little masses on the arbitrary sections parallel to each other is the same.

The difference of the numbers of the little masses passing through the parallel sections F_1 and F_2 is

$$\int\limits_{W} F \, dW$$

which is dependent on the distance s between F_1 and F_2. We have considered the fact that integral of $U(X, \vec{\xi})$ vanishes on the side surface of W while we obtain the formula (10). This result follows from the fact that the side surface of W is composed from the geodesic curves and the normal of the side surface of W is orthogonal to geodesic curves.

If W is a cylindirical domain it is known that

$$\int\limits_{W} \frac{\partial U}{\partial x^n} \, dW = \int\limits_{\partial W} U \, n_{\times}{}^{n}(X) \, dS$$
$$= \int\limits_{F_2} U \, dF_2 - \int\limits_{F_1} U \, dF_1$$

where $n_{\times}{}^{n}(X)$ is the projection of the outher normal of the surface ∂W on $O x^n$.

Since the outher normal of the side surface of ∂W is orthogonal to geodesic curve on the side surface, the projection onto the side surface of the outer normal vanishes, therefore, the integral over this side surface vanishes [4].

Now we present the relation between the theorems of Holditch and Liouville.

The area of F_1 in Liouville Theorem corresponds to the area bounded by γ in Holditch theorem and the area of F_2 corresponds to the area bounded by $\gamma_{\times}$ which is the orbit of X.

The difference between these two areas, in generalized Liouville Theorem, is

$$\int\limits_{W} F \, dW$$

which corresponds to

$$f - f_{\times} = \pi a b$$

in Holditch Theorem.

The independence of $\int_W FdW$, in (10), from the curve γ which is the boundary of F_1, corresponds to the independence of $f-f_\times$ from the curve γ in the Holditch Theorem.

Since the Euclidean metric is ε_{ij}, $= \delta_{ij}$, then the Hamiltonian $H(X, \vec{\xi})$ in generalized Liouville Theorem is given as

$$H(X, \vec{\xi}) = \delta^{ij} \xi_i \xi_j$$

$$= \sum_{i=1}^{n} (\xi_i)^2.$$

The function F (the sources) corresponds to

$$d \left\{ \tfrac{1}{2} (x_1 dx_2 - x_2 dx_1) \right\}$$

in Euclidean metric [1].

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