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SPACELIKE RULED SURFACES IN THE MINKOWSKI 3-SPACE

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ABSTRACT

In this paper, spacelike ruled surfaces, central ponits, curves of striction, developable spacelike ruled surfaces and some theorems related to them are obtained in three dimensional Minkowski space.

1. INTRODUCTION

A surface in 3-dimensional Minkowski space $\mathbb{R}^3_1 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ is called a spacelike surface if the induced metric tensor on the surface is a positive definite Riemannian metric. A ruled surface is a surface swept out by a straight line ℓ moving along a curve α. The various positions of the generating line ℓ are called the rulings of the surface. Such a surface, thus, has a parametrization in the form

$$\varphi(t,v) = a(t) + vZ(t)$$

where we call α the base curve, Z the director vector of ℓ. Alternatively, we may visualize Z as a vector field on α. Frequently, it is necessary to restrict v to some interval, so the rulings may not be the entire straight lines. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surfaces are called skew surfaces. If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main rulings is called central point. The locus of the central points is called the curve of striction. If there is a curve which meets perpendicularly to each one of the rulings, then this curve is called an orthogonal trajectory of a ruled surface (Noel J.H., 1974). If the tangent vector at every point of a given curve in $\mathbb{R}_1^3$ is a spacelike vector, then the given curve is called a spacelike curve (O'Neill, 1983). In $\mathbb{R}_1^3$, we define the exterior product of vectors by $W \wedge V = -(i_V i_W dx \wedge dy \wedge dz)^{\#}$, where i_W denotes the interior product with respect to W and # stands for the operation of raising indices by the metric $dx^2 + dy^2 - dz^2$. Here we choose the mines sign so that $\partial_x \wedge \partial_y = \partial_z$ holds.

2. SPACELIKE RULED SURFACES

Let

$$\alpha: I \to \mathbb{R}_1^3$$
$$t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)),$$

be a differentiable spacelike curve parameterized by arc-lenght in Minkowski 3-space where I is an open interval in $\mathbb{R}$ containing the origin. The tangent vector field of α is denoted by T.

A spacelike straight line,

$$\ell: \mathbb{R} \to \mathbb{R}_1^3$$
$$v \to \ell(v) = (\alpha_1(t) + va_1(t), \alpha_2(t) + va_2(t), \alpha_3(t) + va_3(t))$$

can be chosen so that the director vector of ℓ and the tangent vector of α are linearly independent at every point of the curve α where $a_i(t) \in \mathbb{R}$ for $1 \leq i \leq 3$, are the components of the director vector at a point $\alpha(t)$.

If ℓ moves along α then a ruled surface given by the parametrization

$$\varphi: I \times \mathbb{R} \to \mathbb{R}_1^3$$
$$(t,v) \to \varphi(t,v) = (\alpha_1(t) + va_1(t), \alpha_2(t) + va_2(t), \alpha_3(t) + va_3(t)) \quad (2.1)$$

can be obtained in the Minkowski 3-space. The ruled surface is denoted by M. An orthonormal basis of $\chi(M)$, $\{T, X\}$ can be obtained; thus, $N = T \wedge X$ is a normal of M. Hence, $\{T, X, N\}$ is an orthonormal frame field along α in $\mathbb{R}_1^3$. Let D be Levi-Civita connection on $\mathbb{R}_1^3$. We will obtain variation of this system along α in $\mathbb{R}_1^3$. We define the functions a, b, c by

$$a = \langle D_T T\ ,\ X\rangle = T[\langle T\ ,\ X\rangle] - \langle T\ ,\ D_T X\rangle = -\langle T\ ,\ D_T X\rangle$$
$$b = -\langle D_T T\ ,\ N\rangle = -T[\langle T\ ,\ N\rangle] + \langle T\ ,\ D_T N\rangle = \langle T\ ,\ D_T N\rangle$$
$$c = -\langle D_T T\ ,\ N\rangle = -T[\langle T\ ,\ N\rangle] + \langle X\ ,\ D_T N\rangle = \langle X\ ,\ D_T N\rangle$$

where

$$\begin{aligned} D_T T &= aX + bN \\ D_T X &= -aT + cN \\ D_T N &= bT + cX. \end{aligned} \tag{2.2}$$

The matrix given by

$$B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{bmatrix}$$

is a skew-adjoint matrix in the sense that $B^T = -\varepsilon B \varepsilon$, where

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

For the ruled surface M given by the parametrization (2.1) we have

$$E = \langle \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t} \rangle = (1 - av)^2 - c^2 v^2 \ ,\ F = \langle \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial v} \rangle = 0 \ ,\ G = \langle \frac{\partial \varphi}{\partial v}, \frac{\partial \varphi}{\partial v} \rangle = 1.$$

The induced metric on the ruled surface is positive definite when E is positive. $\min\left\{\frac{1}{a-c}, \frac{1}{a+c}\right\}$ and $\max\left\{\frac{1}{a-c}, \frac{1}{a+c}\right\}$ are roots of E where $a^2 - c^2 = \langle D_T X\ ,\ D_T X\rangle$.

1) If $D_T X$ is the specelike vector field, then
$-\infty < v < \min\left\{\frac{1}{a-c}, \frac{1}{a+c}\right\}$ or $\max\left\{\frac{1}{a-c}, \frac{1}{a+c}\right\} < v < \infty$.

2) If $D_T X$ is the timelike vector filed, then
$\min\left\{\frac{1}{a-c}, \frac{1}{a+c}\right\} < v < \max\left\{\frac{1}{a-c}, \frac{1}{a+c}\right\}$.

3) Let $D_T X$ be the null vector field on $\mathbb{R}^3_1$.
If $a > 0$, then $v < \frac{1}{2a}$, and if $a < 0$, then $v > \frac{1}{2a}$.

Therefore, in all three cases above, the domain of parameter v which is defined in the parametrization of the ruled surface, is not whole of $\mathbb{R}$ but is one of the above intervals. Let J denote the domain of v. If we fix the parameter v in J, then the curve

$$\varphi_v : I\times\{v\} \to M$$
$$(t,v) \to \varphi_v(t,v) = \alpha(t) + vX(t)$$

can be obtained in M. The tangent vector field of this curve is

$$A = (1 - av)T + cvN.$$

Theorem 2.1. Let M be a spacelike ruled surface. The tangent planes along a ruling coincide if and only if $c = 0$.

Proof. It can be seen easily.

Corollary 2.1. The spacelike ruled surface is developable if and only if $c = 0$.

Lemma 2.1. $c = \det(T\ ,\ X\ ,\ D_T X)$ for a spacelike ruled surface.

3. POSITION VECTOR OF A CENTRAL POINT

If the distance, between the central point and the base curve of a spacelike ruled surface which is a skew spacelike surface, is $\bar{u}$ then the position vector $\bar{\alpha}(t)$ can be expressed as

$$\bar{\alpha}(t,\bar{u}) = \alpha(t) + \bar{u}\, X(t)$$

where $\alpha(t)$ is the position vector of the base curve and X(t) is the directed vector belonging to the ruling. The parameter $\bar{u}$ can be expressed in terms of the position vector of the base curve and directed vector of the ruling. Given three preceding rulings of a spacelike ruled surface such that the first one is X(t), and the second one is X(t) + dX(t). Let P, P′ and Q, Q′ be the feet on the rulings of common perpendicular to two preceding rulings. The common perpendicular to X(t) and X(t) + dX(t) is the multiple $X(t) \wedge dX(t)$.

The vector $\overrightarrow{PQ}$ coincides the vector $\overrightarrow{PP'}$ in the limiting position, and $\overrightarrow{PQ}$ will be tangent vector of the curve of striction. Thus, we have $\langle D_T X , \overrightarrow{PQ}\rangle = 0$. Therefore, we get

$$\bar{u} = - \frac{\langle T , D_T X\rangle}{\langle D_T X , D_T X\rangle} = \frac{a}{a^2 - c^2} .$$

Hence, the curve of striction is

$$\bar{\alpha}(t) = \alpha(t) - \frac{\langle T , D_T X\rangle}{\langle D_T X , D_T X\rangle} X(t) \tag{3.1}$$

where $\langle D_T X , D_T X\rangle \neq 0. \bar{u} = \frac{a}{a^2 - c^2}$ is constant since $\langle \frac{d\bar{\alpha}}{dt} , X\rangle = 0$.

Theorem 3.1. The curve of striction $\bar{\alpha}$ does not depend on the choice of the base curve α for the skew spacelike surface.

Proof. Let β be another base curve of the skew spacelike surface; that is, let, for all (t , v),

$$\varphi(t , v) = \alpha(t) + vX(t) = \beta(t) + sX(t)$$

for some function $s = s(v)$. Then, from (3.1) we obtain

$$\bar{\alpha}(t) - \bar{\beta}(t) = \alpha(t) - \beta(t) - \frac{\langle T - \frac{d\beta}{dt} , D_T X\rangle}{\langle D_T X , D_T X\rangle} X(t) = 0$$

since $\langle X , D_T X\rangle = 0$. This proves our claim.

Theorem 3.2. Let M be a skew spacelike surface. $\varphi(t , v_0)$ on the ruling through the point $\alpha(t)$ is central point if and only if $D_T X$ is a normal vector of the tangent plane at $\varphi(t , v_0)$.

Proof. Let $D_T X$ be a normal of the tangent plane at $\varphi(t , v_0)$ on the ruling through $\alpha(t)$. Thus $\langle D_T X , A\rangle = 0$. Hence, we get $v_0 = \frac{a}{a^2 - c^2}$. Therefore, $\varphi(t , v_0)$ is the central point of M.

Conversely, let $\varphi(t , v_0)$ be central point on the ruling through $\alpha(t)$. Then, we have $\langle D_T X , A\rangle = -a + (a^2 - c^2)v = 0$.

On the other hand, $\langle D_T X , X\rangle = 0$. Therefore, $D_T X$ is a normal vector of the tangent plane at $\varphi(t , v_0)$.

$D_T X$ is a timelike vector at the central point as $D_T X$ is a normal vector of the tangent plane at the central point. Thus, $\langle D_T X, D_T X\rangle = a^2 - c^2 < 0$.

Theorem 3.3. The curve of striction

$$\overline{\alpha}(t) = \alpha(t) + \frac{a}{a^2 - c^2} X(t)$$

is a spacelike curve in a skew spacelike surface.

Proof. It can be shown that the tangent vector field of the curve of striction is a spacelike vector field. The tangent vector field of $\overline{\alpha}$ is

$$\frac{d\overline{\alpha}}{dt} = T + \left(\frac{a}{a^2 - c^2}\right) D_T X.$$

Thus, we have $\langle \frac{d\overline{\alpha}}{dt}, \frac{d\overline{\alpha}}{dt}\rangle > 0$ since $c^2 - a^2 > 0$.

Theorem 3.4. Assume that M is a spacelike ruled surface in $\mathbb{R}^3_1$. There exists unique orthogonal trajectory of M through each point of M.

Proof. Let

$$\varphi: I \times J \to \mathbb{R}^3_1$$
$$(t, v) \to \varphi(t, v) = \alpha(t) + vZ(t),$$

be a parametrization of M. An orthogonal trajectory of M is

$$\beta: \tilde{I} \to M$$
$$s \to \beta(s) = \alpha(s) + f(s)Z(s)$$

where $\langle Z(s), Z(s)\rangle = 1$. We may assume that $\tilde{I} \subset I$. Since

$$\langle \frac{d\beta(s)}{ds}, Z(s)\rangle > 0 ,$$

we obtain

$$f(s) = -\int \langle \frac{d\alpha(s)}{ds}, Z(s)\rangle \, ds + h ,$$

where h is a real constant. We get $h = f(s_0) - F(s_0)$ where

$$F(s) = -\int \langle \frac{d\alpha(s)}{ds}, Z(s)\rangle \, ds .$$

Therefore, the orthogonal trajectory of M through P_0 is unique. Thus, we have $\tilde{I} = I$ since the orthogonal trajectory of M meets each ruling.

Theorem 3.5. Suppose that M is skew spacelike surface. The longest distance between two rulings is measured only by the curve of striction which is one of the orthogonal trajectories through these two rulings.

Proof. Fixing two rulings say for $t_1 < t_2$, we compute the length j(v) of an orthogonal trajectory between these two rulings by

$$j(v) = \int_{t_1}^{t_2} \|A\| \, dt = \int_{t_1}^{t_2} \sqrt{|\langle A , A\rangle|} \, dt = \int_{t_1}^{t_2} \left[(a^2 - c^2)v^2 - 2av + 1\right]^{1/2} dt .$$

Let us find the value of s which maximizes j(v), and we get $\frac{\partial j(v)}{\partial v} = 0$. Thus, we have $v = \frac{a}{a^2 - c^2}$. This completes the proof.

Example 1. (The helicoid of the 1st kind). This is a spacelike ruled surface parametrized by,

$$\varphi(t , v) = \left(\left(\frac{\kappa}{\kappa^2 - \tau^2} - v\right)\cos\sqrt{\kappa^2 - \tau^2}\, t , \left(\frac{\kappa}{\kappa^2 - \tau^2} - v\right)\sin\sqrt{\kappa^2 - \tau^2}\, t , \frac{\tau}{\sqrt{\kappa^2 - \tau^2}}\, t\right),$$

(Waestijne, 1990), where κ and τ are the curvature and the torsion of α respectively, and $|\kappa| > |\tau|$. The base curve α is a spacelike curve since $\langle\frac{d\alpha}{dt} , \frac{d\alpha}{dt}\rangle = 1$, and each ruling is a spacelike line.

Now, $v < \min\left\{\frac{-1}{\tau + \kappa}, \frac{1}{\tau - \kappa}\right\}$ or $v > \max\left\{\frac{-1}{\tau + \kappa}, \frac{1}{\tau - \kappa}\right\}$ since $D_T X$ is a specelike vector field. Furthermore, $\det(T , X , D_T X) = \tau$. The leicoid of the 1st kind is developable if and only if $\tau = 0$ (Fig. (1)).

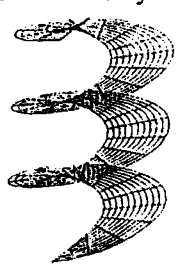

Fig. (1)

Example 2. (The helicoid of the 2nd kind). This is a spacelike ruled surface parametrized by,

$$\varphi(t, v) = \left(\left(\frac{\kappa}{\kappa^2 - \tau^2} - v \right) \mathrm{ch}\sqrt{\kappa^2 - \tau^2}\, t, \frac{\tau}{\sqrt{\kappa^2 - \tau^2}}\, t, \left(\frac{\kappa}{\kappa^2 - \tau^2} - v \right) \mathrm{sh}\sqrt{\kappa^2 - \tau^2}\, t \right),$$

(Woestijne, 1990), where $|\tau| > |\kappa|$. The base curve α is a spacelike curve, and each ruling is a spacelike line. Now,

$$\min\left\{\frac{-1}{\tau - \kappa}, \frac{-1}{\tau + \kappa}\right\} < v < \max\left\{\frac{-1}{\tau - \kappa}, \frac{-1}{\tau + \kappa}\right\}$$

since $D_T X$ is a timelike vector field. The line of striction is

$$\bar{\alpha}(t) = \alpha(t) + \frac{\kappa}{\kappa^2 - \tau^2} X(t),$$

and $\bar{\alpha}(t)$ is a spacelike curve. Furthermore, $\det(T, X, D_T X) = -\tau$. The helicoid of the 2nd kind is developable if and only if $\tau = 0$ (Fig. (2)).

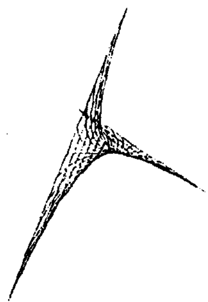

Fig. (2)

Example 3. (The conjugate surface of Enneper of the 2nd kind). This is a spacelike ruled surface parametrized by,

$$\varphi(t, v) = \left(\frac{\kappa\tau^2}{2} + v, \frac{-\tau^2 t^3}{6} + t - \kappa v t, \frac{\kappa\tau t^3}{6} + \tau v t \right),$$

(Woestijne, 1990), where $|\kappa| = |\tau| \neq 0$. The base curve α is a spacelike curve, and each ruling is a spacelike line. Now

$$v < \frac{1}{2\kappa} \quad \text{If} \quad \kappa > 0$$

$$v > \frac{1}{2\kappa} \quad \text{If} \quad \kappa < 0,$$

since $D_T X$ is the null vector field. Furthermore, $\det(T , X , D_T X) = -\tau$. The conjugate surface of Enneper of the 2nd kind is developable if and only if $\tau = 0$ (Fig. (3)).

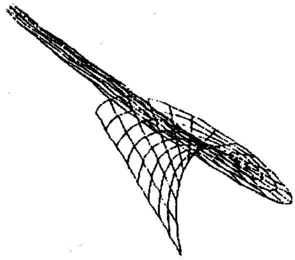

Fig. (3)

Example 4. This is a spacelike ruled surface paremetrized by,

$$\varphi(t , v) = \alpha(t) + vX(t) = (0 , t , 0) + v(t , 0 , 0),$$

(Woestijne, 1990). The base curve is a spacelike curve, and each ruling is a spacelike line. This ruled surface is developable.

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