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UNIVALENT HARMONIC MAPPINGS

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ABSTRACT

A family of univalent harmonic functions is studied from the point of geometric function theory. This class consists of mappings of the open unit disk onto the entire complex plane except for two infinite slits along the real axis with a normalization at the origin. Extreme points are determined, and sharp estimates for Fourier coefficients and distortion theorems are given.

1. INTRODUCTION

Clunie and Sheil-Small [1] studied the class S_H of all harmonic, complex-valued, sense preserving univalent mappings f defined on the open unit disk U which are normalized by $f(0)=0$, $f_z(0)=1$. Such functions admit the representation $f=h+\overline{g}$ where $h(z) = z+a_2z^2+\dots$ and $g(z)=b_1z+b_2z^2+\dots$ are analytic in U. f is locally one-to-one and sense preserving if and only if $|g'(z)| < |h'(z)|$ for z is in U. This implies that $|b_1| < 1$. Therefore $f_0 = (f-\overline{b_1 f})/(1-|b_1|^2)$ is also in S_H and one may restrict attention to the subclass $S_H^0 = \{f \in S_H: f_{\bar z}(0) = 0\}$.

If $f = u+iv$ is harmonic in U with $f(0)=0$, we let F and G be analytic in U and satisfy $F(0)=G(0)=0$, $u=\text{Re } F$ and $v=\text{Re } G$. If we let $h=(F+iG)/2$ and $g=(F-iG)/2$ then h and g are analytic in U and $f=h+\overline{g}$.

In contrast to conformal mappings, harmonic mappings are not essentially determined by their image domains. Therefore, it is natural to study the class $S_H(U,D_\phi)$ of harmonic, sense preserving and univalent mappings of U onto another domain $D_\phi=\mathbb{C}-(-\infty,a_\phi]\cup[b_\phi,+\infty)$ normalized by $f(0) = 0$, $f_{\bar z}(0) = 0$ and $f_z(0) = 1$, where ϕ is a fixed parameter $(0<\phi<\pi)$, and the constants a_ϕ, b_ϕ $(a_\phi < 0 < b_\phi)$ are determined as in Theorem 1. If $\phi\to 0$, our results will give those of Livingston [2].

2. THE CLASS $S_H(U,D_\phi)$

Let P be a class of p(z), which are analytic in U with p(0)=1 and Re p(z)>0 for z in U.

Lemma 1. If p(z) is in P, then, for $0<\phi<\pi$,

$$-\frac{1}{2(1+\cos\phi)}\left(1+\frac{\phi}{\sin\phi}\right) \le \mathrm{Re}\int_0^{-1}\frac{(1-\zeta^2)p(\zeta)d\zeta}{(1-2\cos\phi\zeta+\zeta^2)^2} \le \frac{1}{2(1-\cos\phi)}\left(1-\frac{\phi}{\sin\phi}\right) \tag{1}$$

$$\frac{1}{2(1+\cos\phi)}\left(\frac{\pi-\phi}{\sin\phi}-1\right) \le \mathrm{Re}\int_0^{1}\frac{(1-\zeta^2)p(\zeta)d\zeta}{(1-2\cos\phi\zeta+\zeta^2)^2} \le \frac{1}{2(1-\cos\phi)}\left(\frac{\pi-\phi}{\sin\phi}+1\right) \tag{2}$$

$$\frac{(2\phi-\pi)\cos\phi-2\sin\phi+\pi}{2\sin^3\phi} \le \mathrm{Re}\int_{-1}^{1}\frac{(1-\zeta^2)p(\zeta)d\zeta}{(1-2\cos\phi\zeta+\zeta^2)^2} \le \frac{(\pi-2\phi)\cos\phi+2\sin\phi+\pi}{2\sin^3\phi} \tag{3}$$

Proof. We set $w=e^{i\phi}$, $0<\phi<\pi$. We estimate the integral

$$I = \mathrm{Re}\int_0^{-1}\frac{(1-\zeta^2)p(\zeta)d\zeta}{(1-2\cos\phi\zeta+\zeta^2)^2} = -\int_0^{1}\frac{1-t^2}{(1+wt)^2(1+\overline{w}t)^2}\,\mathrm{Re}\ p(-t)\ dt. \tag{4}$$

It is well known that for $-1<t<1$

$$(1-|t|)/(1+|t|)\le \mathrm{Re}\ p(t)\le(1+|t|)/(1-|t|). \tag{5}$$

Substituting (5) into (4), we obtain

$$-\int_0^{1}\frac{1-t^2}{(1+wt)^2(1+\overline{w}t)^2}\frac{1+t}{1-t}\,dt \le I \le -\int_0^{1}\frac{1-t^2}{(1+wt)^2(1+\overline{w}t)^2}\frac{1-t}{1+t}\,dt.$$

Since

$$\log\left(\frac{1-e^{i\phi}}{1-e^{-i\phi}}\right) = i(\phi-\pi) \quad \text{and} \quad \log\left(\frac{1+e^{i\phi}}{1+e^{-i\phi}}\right) = i\phi \ ,$$

inequality (1) is readily obtained.

(2) can be proved in the same way. From (1) and (2), we have (3).

Remark. The expression on the left hand side of (1) tends to -1/2 as $\phi\to0^+$ while the expression on the right hand side of (1) tends to -1/6 as $\phi\to0^+$. These bounds have been given by Livingston [2,Lemma 1]. Moreover, the upper and lower bounds in (3) have a minimum for $\phi=\pi/2$.

We now let $\mathfrak{F}_\phi$ be the class of functions f which have the form

$$f(z) = \mathrm{Re}\int_0^z \frac{1-\zeta^2}{(1-2\cos\phi\ \zeta+\zeta^2)^2}\, p(\zeta)d\zeta + i\ \mathrm{Im}\ \frac{z}{1-2\cos\phi\ z+z^2} \tag{6}$$

where $p\in P$ and ϕ is a fixed parameter in the interval $(0,\pi)$.

Theorem 1. If $f\in\mathfrak{F}_\phi$, then f is harmonic, sense preserving and univalent in U and f(U) is convex in the direction of the real axis with $f(U)\subset D_\phi$.

Proof. Let $f=h+\bar{g}=\mathrm{Re}\ F+i\mathrm{Re}\ G$. Then, we have from (6) that

$$F(z) = \int_0^z \frac{1-\zeta^2}{(1-2\cos\phi\zeta+\zeta^2)^2}\, p(\zeta)d\zeta \quad \text{and} \quad G(z) = \frac{-i\ z}{1-2\cos\phi\ z+z^2} \tag{7}$$

for z in U. Since $F'(z)/iG'(z)=p(z)$ and

$$\frac{g'(z)}{h'(z)} = \frac{F'(z)-iG'(z)}{F'(z)+iG'(z)} = \frac{p(z)-1}{p(z)+1}$$

it follows that $|g'(z)|<|h'(z)|$. Thus, f is locally one to one and sense preserving. Also,

$$h(z)-g(z) = iG(z) = \frac{z}{1-2\cos\phi\ z+z^2}$$

is convex in the direction of the real axis. By a theorem of Clunie and Sheil-Small [1, Theorem 5.3], f is univalent and f(U) is convex in the direction of the real axis.

Moreover, f(z) is real if and only if z is real. Since Re $p(z)>0$, it follows that $f(r)=\text{Re } F(r)$ is increasing on (-1,1) and f(r) is bounded on (-1,1) by the Lemma 1 for a fixed ϕ, $0<\phi<\pi$. Thus, $\lim_{r\to -1^+} f(r)$ and $\lim_{r\to 1^-} f(r)$ exists and equals to a_ϕ and b_ϕ, respectively. Thus, f(U) does not contain the interval $(-\infty,a_\phi]\cup[b_\phi,\infty)$. Therefore, $f(U)\subset D_\phi$.

Theorem 2. $S_H(U,D_\phi)\subset\mathfrak{F}_\phi$.

Proof. Let $f\in S_H(U,D_\phi)$. Since $f(U)=D_\phi$ is convex in the direction of real axis for a fixed ϕ, by the theorem, given by Clunie and Sheil-Small [1,Theorem 5.3], h-g=iG is univalent and convex in the direction of real axis.

Let $h(z)=z+a_2z^2+...$ and $g(z)=b_2z^2+...$ Then, $iG(z)=h(z)-g(z)=z+...$ Since $f(U)=D_\phi$, Re G(z)=Im f(z) is 0 on the boundary of U. Since G is convex in the direction of the imaginary axis, it follows that G(U) is $\mathbb{C}$ slit along two infinite rays on the real axis for $\phi\in(0,\pi)$. Also, since $iG(0)=iG(0)-1=0$, it follows that iG(z) is a member of the class S of functions f which are analytic and univalent in U and normalized by $f(0)=f'(0)-1=0$. Thus, there is a fixed ϕ, $0<\phi<\pi$, such that

$$iG(z) \prec k_\phi(z) = \frac{z}{1-2\cos\phi\ z+z^2}$$

where $\prec$ denotes subordination. Since $k_\phi\in S$, it follows that $iG=k_\phi$. Hence, Im f(r)=Re G(r)=0 for $-1<r<1$.

Now, if $f=h+\overline{g}$, then $h'-g'=iG'$ and

$$\frac{h'+g'}{h'-g'} = \frac{1+g'/h'}{1-g'/h'} .$$

Since $|g'(z)|<|h'(z)|$, for z in U, it follows that

$$(h'+g')/(h'-g') = p,$$

where $p \in P$. Thus, $h'+g' = (h'-g')p = iG'p$

$$F(z) = h(z)+g(z) = \int_0^z iG'(\zeta)p(\zeta)d\zeta = \int_0^z \frac{1-\zeta^2}{(1-2\cos\phi\zeta+\zeta^2)^2} p(\zeta)d\zeta .$$

Thus $f(z)=\mathrm{Re}\, F(z)+i\mathrm{Re}\, G(z)$ belongs to $\mathfrak{F}_\phi$.

Theorem 3. $\overline{S_H(U,D_\phi)}=\mathfrak{F}_\phi$.

Proof. Let $f\in\mathfrak{F}_\phi$ have the form (4), and let r_n be a sequence with $0<r_n<1$ and $\lim r_n=1$. Let $p_n(z)=p(r_nz)$, and denote by $f_n(z)$ the function obtained from (4) by replacing $p(z)$ with $p_n(z)$. We claim that f_n is in $S_H(U,D_\phi)$. To see this, let

$$F_n(z) = \int_0^z \frac{1-\zeta^2}{(1-2\cos\phi\zeta+\zeta^2)^2} p(\zeta)d\zeta .$$

There exists $\delta_i>0$, $i=1,2$, so that we may write for $|z-w_i|<\delta_i$

$$p_n(z)=p_n(w_i)+p'_n(w_i)(z-w_i)+[p''_n(w_i)/2!](z-w_i)^2+...$$

where $w_1=e^{i\phi}$ and $w_2=e^{-i\phi}$. Then, for $|z-w_1|<\delta_1$,

$$F'_n(z) = \left[\frac{1}{w_1^2-1}\frac{1}{(z-w_2)^2} + \frac{1}{w_2^2-1}\frac{1}{(z-w_1)^2}\right] p_n(z)$$

$$= \left[\frac{p_n(w_1)}{(w_2^2-1)(z-w_1)^2} + \frac{p'_n(w_1)}{(w_2^2-1)(z-w_1)} + q_1(z)\right],$$

where $q_1(z)$ is analytic in $|z-w_1|<\delta_1$. Let $D_i=\{z:|z-w_i|<\delta_i\}\cap U$, $i=1,2$. If $1-\delta_i<c_i<1$, then, for z in D_i,

$$F_n(z) - F_n(c_i) = \int_{c_i}^z F'_n(\zeta)d\zeta . \tag{8}$$

where the path of integration is in D_i. Equation (8) gives for $z\in D_1$.

$$F_n(z) = \left[\frac{p_n(w_1)}{(1-w_2^2)(z-w_1)} - \frac{p'_n(w_1)}{1-w_2^2} + \log(z-w_1) + q(z)\right],$$

where

$$q(z) = \sum_{j=0}^{\infty} \lambda_j(z-w_1)^j + \sum_{j=0}^{\infty} \mu_j(z-w_2)^j + \frac{b}{z-w_2} + a \log(z-w_2)$$

is analytic in D_1 and $\arg(z-w_1) \in (0,\pi)$. Thus, for $z \in D_1$, F_n has the form

$$F_n(z) = \left[\frac{k}{z-w_1} + m \log(z-w_1)\right]$$

and then

$$\begin{aligned} \text{Re } f_n(z) &= \text{Re } F_n(z) \\ &= \left[\text{Re } \frac{k}{z-w_1} + \text{Re}(m) \ln|z-w_1| - \text{Im}(m)\arg(z-w_1) + \text{Re } q(z)\right] \end{aligned}$$

Now, we wish to prove that f_n cannot have a nonreal finite cluster point at $z=w_1$. To see this, suppose that $z_j=w_1+\rho_j e^{i\alpha_j}$ is in U with $\rho_j>0$ and $\lim \rho_j=0$. We claim that $|\text{Re } f_n(z_j)| \to \infty$ as $n \to \infty$. Indeed,

$$\text{Re } f_n(z_j) = \left|\frac{\text{Re}(ke^{-i\alpha_j}) + \rho_j \text{Re}(m)\ln(\rho_j)}{\rho_j} - \text{Im}(m)\arg(z_j-w_1) + \text{Re } q(z_j)\right|$$

approaches to $+\infty$ as n approaches to $+\infty$. Similarly, we have the same argument for D_2. Thus, f_n has no finite nonreal cluster points at $z=w_1$ and $z=w_2$. At all other points of $|z|=1$, the finite cluster points of f_n are real. Since $f_n(U) \subset D_\phi$, and

$$\lim_{r \to -1^+} f_n(r) = a_\phi \quad , \quad \lim_{r \to 1^-} f_n(r) = b_\phi$$

it follows that $f_n(U)=D_\phi$ for a fixed ϕ.

Thus, f_n is in $S_H(U,D_\phi)$ and hence, f is in $\overline{S_H(U,D_\phi)}$. Since $\mathfrak{F}_\phi$ is closed under uniform limits on compact subsets of U, it follows that $\mathfrak{F}_\phi = S_H(U,D_\phi)$.

3. EXTREME POINTS OF $\mathfrak{F}_\phi$

If $p\in P$, then it is known that

$$p(z) = \int_{|\eta|=1} \frac{1+\eta z}{1-\eta z}\, d\mu(\eta)\ , \tag{9}$$

where μ is a probability measure on $X=\{\eta:|\eta|=1\}$. Thus, if f is in $\mathfrak{F}_\phi$, there is a probability measure μ on X such that

$$f(z) = \left[\mathrm{Re} \int_{|\eta|=1} k_\phi(z,\eta)d\mu(\eta)+i\mathrm{Im}\ k_\phi(z)\right]$$

and

$$k_\phi(z,\eta) = \int_0^z \frac{(1-\zeta^2)(1+\eta\zeta)}{(1-2\cos\phi\ \zeta+\zeta^2)^2(1-\eta\zeta)}\, d\zeta \tag{10}$$

$$= \begin{cases} A(w,\eta)\log(1-z\bar{w})+A(\bar{w},\eta)\log(1-wz)+B(w,\eta)\dfrac{\bar{w}^2 z}{1-\bar{w}z} + \\ \qquad B(\bar{w},\eta)\dfrac{w^2 z}{1-wz} + C(w,\eta)\log(1-\eta z) \ ; \ \text{if } \eta \neq w,\bar{w} \\ \dfrac{i}{4\sin^3\phi}\log\left(\dfrac{1-wz}{1-\bar{w}z}\right) - \dfrac{\cos\phi\bar{w}z}{2\sin^2\phi(1-\bar{w}z)} - \dfrac{iwz}{2\sin\phi(1-wz)^2} \ ; \ \text{if } \eta = w \\ \dfrac{i}{4\sin^3\phi}\log\left(\dfrac{1-wz}{1-\bar{w}z}\right) - \dfrac{\cos\phi wz}{2\sin^2\phi(1-wz)} + \dfrac{i\bar{w}z}{2\sin\phi(1-\bar{w}z)^2} \ ; \ \text{if } \eta = \bar{w} \end{cases}$$

for $\phi\in(0,\pi)$

$$w=e^{i\phi}\ ,\ A(w,\eta) = \frac{2\eta w^2}{(1-\eta w)^2(1-w^2)}\ ,\ B(w,\eta) = \frac{(1+\eta w)w^2}{(1-\eta w)(1-w^2)}$$

and

$$C(w,\eta) = \frac{2\eta(1-\eta^2)}{(1-\eta w)^2(1-\eta\bar{w})^2}\ .$$

The extreme points of $\mathfrak{F}_\phi$ are readily obtained by making use of the consequence observed by Szapiel [3].

Lemma 2. [3]. Suppose X is a convex linear Hausdorff space, $\Phi: X \to \mathbb{C}$ is homogeneous, $c \in \mathbb{C}\setminus\{0\}$ and A is a compact convex subset of $\Phi^{-1}(c)$. Let $\psi: A \to R$ be affine continuous with $0 \notin \psi(A)$ and let $B=\{a/\psi(a): a \in A\}$. Then

1) B is compact convex,

2) The map $a \to a/\psi(a)$ is a homeomorphism of A onto B,

3) $E_B=\{a/\psi(a): a \in E_A\}$, where E_P shows the set of all extreme points of P.

Theorem 4. The extreme points of $\mathfrak{F}_\phi$ are

$$f_\eta(z)=[\mathrm{Re}\ k_\phi(z,\eta)+i\mathrm{Im}\ k_\phi(z)]\ ,\ |\eta|=1.$$

Proof. We apply Lemma 2 with

$$Q_p(z) = \mathrm{Re}\left[\int_0^z \frac{(1-\zeta^2)p(\zeta)}{(1-2\cos\phi\ \zeta+\zeta^2)^2}\, d\zeta\right] + \mathrm{Im}\left[\frac{z}{1-2\cos\phi\ z+z^2}\right]$$

$$A=\{Q_p: p \in P\}\ ,\ \Phi(f)=f_z(0)=1\ ,\ c=1 \text{ and } \psi(Q_p)=1$$

Then $\mathfrak{F}_\phi=B$ is convex. The map $Q_p \to p$ is a linear homeomorphism between A and P. $E_P=\{(1+\eta z)/(1-\eta z): |\eta|=1\}$. Thus, the proof of theorem is completed.

4. APPLICATIONS

In this section, we will use our knowledge of extreme points to solve some extremal problems on $\overline{S_H(U,D_\phi)}$.

Theorem 5. Let $f=h+\bar{g} \in \overline{S_H(U,D_\phi)}$. If $h(z) = z+\sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$, then, for $0<\phi<\pi$,

$$|a_n| \le \frac{1}{n} \sum_{k=1}^{n} k \frac{|\sin(k\phi)|}{\sin\phi} < \frac{(n+1)(2n+1)}{6}, \tag{11}$$

$$|b_n| \le \frac{1}{n} \sum_{k=1}^{n-1} k \frac{|\sin(k\phi)|}{\sin\phi} < \frac{(n-1)(2n-1)}{6} \tag{12}$$

and

$$\big||a_n| - |b_n|\big| \le \frac{|\sin(n\phi)|}{\sin\phi} < n \ . \qquad (13)$$

Equality in (11), (12) and (13) occours for the function

$$f(z)=[\mathrm{Re}\ k_\phi(z,e^{\pm i\phi})+i\mathrm{Im}\ k_\phi(z)] \ .$$

Proof. In order to prove validity of (11), (12) and (13), we will make use of the extreme points of $\overline{S_H(U,D_\phi)}$. Let $f_\eta(z)=[\mathrm{Re}\ k_\phi(z,\eta)+i\mathrm{Im}\ k_\phi(z)]$. Also, $F(z)=k_\phi(z,\eta)$ and $G(z)=-iz/(1-2\cos\phi\ z+z^2)$. Thus

$$h(z) = \frac{1}{2}\,[F(z)+iG(z)] = \frac{1}{2}\,[k_\phi(z,\eta)+k_\phi(z)] = z+\sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = \frac{1}{2}\,[F(z)-iG(z)] = \frac{1}{2}\,[k_\phi(z,\eta)-k_\phi(z)] = z+\sum_{n=2}^{\infty} b_n z^n$$

if $\eta\neq e^{\pm i\phi}$, for $w=e^{i\phi}$, then we have

$$h(z) = \frac{1}{2}\Bigg[\frac{-2\eta w^2}{(1-\eta w)^2(1-w^2)}\sum_{n=1}^{\infty}\frac{w^{-n}}{n}z^n - \frac{2\eta w^{-2}}{(1-\eta w^{-1})^2(1-w^{-2})}\sum_{n=1}^{\infty}\frac{w^{n}}{n}z^n$$
$$+\frac{(1+\eta w)w}{(1-\eta w)(1-w^2)}\sum_{n=1}^{\infty} w^{-n}z^n + \frac{(1+\eta w^{-1})w^{-1}}{(1-\eta w^{-1})(1-w^{-2})}\sum_{n=1}^{\infty} w^{n}z^n$$
$$-\frac{2\eta(1-\eta^2)}{(1-\eta w)^2(1-\eta w^{-1})^2}\sum_{n=1}^{\infty}\frac{\eta^n}{n}z^n + \sum_{n=1}^{\infty}\frac{w^n-w^{-n}}{w-w^{-1}}z^n\Bigg]$$

Therefore,

$$a_n = \frac{1}{2}\Bigg[\frac{-2\eta w^{2-n}}{n(1-\eta w)^2(1-w^2)} - \frac{2\eta w^{n-2}}{(1-\eta w^{-1})^2(1-w^{-2})} + \frac{(1+\eta w)w^{1-n}}{(1-\eta w)(1-w^2)}$$
$$+\frac{(1+\eta w^{-1})w^{n-1}}{(1-\eta w^{-1})(1-w^{-2})} - \frac{2\eta(1-\eta^2)\eta^n}{n(1-\eta w)^2(1-w^{-1})^2} + \frac{w^n-w^{-n}}{w-w^{-1}}\Bigg]$$

$$= \frac{1}{2}\Bigg(\frac{-2\eta[w^{n-1}-w^{1-n}-2\eta(w^n-w^{-n})+\eta^2(w^{n+1}-w^{-n-1})+(1-\eta^2)\eta^n(w-w^{-1})]}{n(w-w^{-1})(1-\eta w)^2(1-\eta w^{-1})^2}$$
$$+\frac{2(w^n-w^{-n})-2\eta(w^{n+1}w^{-n-1})}{(w-w^{-1})(1-\eta w)(1-\eta w^{-1})}\Bigg)$$

$$= \frac{\eta}{n}\Bigg[\frac{-\sum_{k=1}^{n-1} w^{n-2k}+2\eta\sum_{k=1}^{n} w^{n-2k+1}-\eta^2\sum_{k=0}^{n} w^{n-2k}-\eta^n+\eta^{n+2}}{(1-\eta w)^2(1-\eta w^{-1})^2} + \frac{n\left(\sum_{k=1}^{n} w^{n-2k+1}-\eta\sum_{k=0}^{n} w^{n-2k}\right)}{\eta(1-\eta w)(1-\eta w^{-1})}\Bigg]$$

$$= \frac{1}{n}\left[\frac{\sum_{k=1}^{n}\eta^{k+1}(w^{n-k}+w^{k-n})-\eta\sum_{k=1}^{n-1}w^{n-2k}+n\sum_{k=1}^{n}w^{n-2k+1}-\eta n\sum_{k=0}^{n}w^{n-2k}}{(1-\eta w)(1-\eta w^{-1})}\right]$$

$$= \frac{1}{n}\Big[\eta^{n-1}+2\eta^{n-2}(w+w^{-1})+3\eta^{n-3}(w^{2}+w^{-2}+1)+ \ldots+ +n(w^{n-1}+w^{1-n}+w^{n-3}+w^{3-n}+\ldots+\lambda)\Big]$$

where $\lambda=w+w^{-1}$ if n is even, $\lambda=1$ if n is odd. And so for n=2,3,...

$$a_n = \frac{1}{n}(w+w^{-1})^{-1}\sum_{k=0}^{n-1}(n-k)\eta^{k}(w^{n-k}-w^{k-n}) = \frac{1}{n\ \sin\phi}\sum_{m=1}^{n} m\ \eta^{n-m}\sin(m\phi).$$

Thus,

$$|a_n| \leq \frac{1}{n\ \sin\phi}\sum_{m=1}^{n} m|\sin(m\phi)| < \frac{(n+1)(2n+1)}{6}$$

with equality for $\eta=e^{\pm i\phi}$.

Similarly, for n=2,3,..., we have

$$b_n = \frac{1}{n}(w-w^{-1})^{-1}\sum_{k=1}^{n-1}(n-k)\eta^{k}(w^{n-k}-w^{k-n}) = \frac{1}{n\ \sin\phi}\sum_{m=1}^{n-1} m\ \eta^{n-m}\sin(m\phi)$$

from which

$$|b_n| \leq \frac{1}{n\ \sin\phi}\sum_{m=1}^{n-1} m|\sin(m\phi)| < \frac{(n-1)(2n-1)}{6}$$

with equality for $\eta=e^{\pm i\phi}$.

Remark. If $\phi\to 0$, our results in the Theorem 5 give those of Livingston [2, Theorem 5].

Theorem 6. If $f=h+\bar{g}$ is in $S_H(U,D_\phi)$, then

$$|f_z(z)| \leq \frac{1+|z|^2}{(1-|z|)^5} \quad \text{and} \quad |f_{\bar{z}}(z)| \leq \frac{|z|\ (1+|z|^2)}{(1-|z|)^5} \tag{14}$$

Equality in (14) occurs for the functions

$$f(z)=[\mathrm{Re}\ k_\phi(z,e^{\pm i\phi})+i\mathrm{Im}\ k_\phi(z)]$$

Proof. We need only to consider extreme points $f_\eta(z)$. In this case for $\eta\neq e^{i\phi}$, $w=e^{i\phi}$, it is concluded that

$$\begin{aligned} h(z) &= \frac{1}{2}\,[k_\phi(z,\eta)+k_\phi(z)] \\ &= \frac{1}{2}\Bigg[A(w,\eta)\log(1-w^{-1}z)+A(w^{-1},\eta)\log(1-wz)+B(w,\eta)\,\frac{w^{-2}z}{1-w^{-1}z} \\ &\quad + B(w^{-1},\eta)\,\frac{w^{2}z}{1-wz}+C(w,\eta)\log(1-\eta z)+\frac{z}{1-(w+w^{-1})z+z^{2}}\Bigg]. \end{aligned}$$

After having straightforward computations, we have

$$h'(z) = \frac{1-z^2}{(1-w^{-1}z)^2\,(1-wz)^2\,(1-\eta z)}$$

$$|h'(z)| = \left|\frac{1-z^2}{(1-w^{-1}z)^2\,(1-wz)^2\,(1-\eta z)}\right|$$

and

$$|h'(z)| \leq \frac{1}{1-|z|}\left|\frac{1-z^2}{(1-w^{-1}z)^2\,(1-wz)^2}\right| \leq \frac{1+|z|^2}{(1-|z|)^5}$$

Similarly, for $\eta\neq e^{\pm i\phi}$, we obtain

$$g(z) = \frac{1}{2}\,[k_\phi(z,\eta)-k_\phi(z)]\ ,\ g'(z) = \frac{z(1-z^2)\eta}{(1-w^{-1}z)^2\,(1-wz)^2\,(1-\eta z)}$$

and

$$|g'(z)| \leq \frac{1}{1-|z|}\left|\frac{z(1-z^2)}{(1-w^{-1}z)^2\,(1-wz)^2}\right| \leq \frac{|z|\,(1+|z|^2)}{(1-|z|)^5}\ .$$

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